

Financial fragility and global dynamics

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Abstract

This paper deals with a simple model of financial fluctuations, where a crucial role is played by the dynamic interaction between aggregate *current* and *intertemporal* financial ratios. The model results in a 4D discrete-time dynamical system – capable of generating complex dynamics – which is analyzed by means of both analytical tools, such as local stability analysis and bifurcation theory, and numerical simulations. The behavior of the model is studied for different parameter regimes. We show that its dynamic behavior is very sensitive to the parameters that represent (1) the speed of adjustment of the desired current financial ratio towards a safe level of the intertemporal one and (2) the intensity with which aggregate current financial decisions affect future financial constraints. In particular, different parameter regimes are identified, giving rise to two different “routes” to complexity, one leading to chaotic dynamics, the other to a coexistence of attractors and path-dependence.

1 Introduction

Business cycles are complex phenomena where many variables have an impact on the overall behavior of the aggregate variables of the economy. Casual empiricism and a long tradition (originated, among others by Frisch, Keynes and Minsky) suggest that the complex behavior of business cycles crucially depends on the fluctuations of the financial fragility of financial units. This point has been explained and modelled in different ways. This paper follows Vercelli [1] and Sordi and Vercelli [2] in rooting the complexity of financial fluctuations in the feedback between the current and the

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intertemporal financial conditions. The purpose of this paper is to show that a very simple model of financial fluctuations is sufficient to mimic a wide variety of the complex behaviors revealed by the empirical evidence concerning financial fluctuations. This suggests that the mechanism depicted by the model is likely to capture a crucial determinant of business cycles.

The paper is organized as follows. In Section 2 we present the model of aggregate financial fluctuations and reduce it to a nonlinear discrete-time dynamical system. Section 3 discusses the steady state and the local dynamics of the model by means of qualitative tools, such as local stability analysis and bifurcation theory. Section 4 uses numerical simulation and graphical analysis to discuss the global dynamics of the model. In particular, this section studies its behavior under those parameter regimes for which the steady state is unstable (Sections 4.1 and 4.2), and develops an analysis of the basin of attraction of the steady state in those regimes where it is only locally, but not globally stable (Section 4.3). Section 5 concludes. A mathematical appendix at the end of the paper contains some technical details.

2 The model

The financial units operating in the economy are characterized in each period t by financial outflows – which correspond to purchases of goods and services, as well as payment of interest and repayment of principal on outstanding debt – and financial inflows which correspond to sales of goods and services and new loans. We call the *current financial ratio* the ratio between the financial outflows and inflows that occurred in a certain period of time. By aggregating the outflows and inflows of all financial units we obtain the aggregate outflows of the private sector E_t^{pr} , and the aggregate inflows of the same sector Y_t . Their ratio designates the current financial ratio for the private sector of the entire economy:

$$C_t = \frac{E_t^{pr}}{Y_t}$$

So far we have considered realized magnitudes. The current financial ratio, however, does not necessarily correspond to its desired value – which we denote by K_t . We introduce in the model the distinction between realized and desired financial ratio (absent in the original model developed in [1] and [2]) as we believe that both magnitudes play a crucial role in explaining the dynamic behavior of the financial units of the economy and, therefore, also of the aggregate variables that we are going to analyze.

We assume that financial units at the end of period $t - 1$ decide their financial outflows in period t on the basis of their desired financial ratio for

that period, with a lag of one period between realized inflows and outflows:

$$E_t^{pr} = K_t Y_{t-1}$$

Changes in the current financial position, in turn, affect the financial constraints faced by the units in the future. To take account of this, a crucial variable which captures the viability of the financial side of the economy is the aggregate *intertemporal financial ratio*, K_t^* , defined as the ratio between the sum of discounted expected future financial outflows and the sum of discounted expected future inflows:

$$K_t^* = \frac{\sum_{s=0}^m \mathbb{E}_{t-1} [E_{t+s}^{pr}] / (1+r)^s}{\sum_{s=0}^m \mathbb{E}_{t-1} [Y_{t+s}] / (1+r)^s}$$

where $\mathbb{E}_{t-1}[\cdot]$ denotes the conditional expectation operator, based upon information available at the end of period $t-1$ (beginning of period t),¹ m is the time horizon and r is the exogenously given interest rate. The intertemporal financial ratio may thus vary over time as the (common) expectations of future financial inflows and outflows vary. In turn, expectations about future financial inflows/outflows are affected by new information contained in recent realizations of the variables. Since the decisions makers are characterized by bounded rationality, they revise their expectations according to an adaptive mechanism: if realized outflows (inflows), in period t are higher or lower than expected for that period, expectations of future outflows (inflows) are revised according to the sign of the deviation. The simplest way to incorporate this idea into the model is to assume that the intertemporal financial ratio varies through the following mechanism, such that

$$K_{t+1}^* = K_t^* + \beta \left(\frac{E_t^{pr}}{Y_t} - K_t^* \right)$$

where $\beta > 0$ can be interpreted as the intensity with which the current financial exposure affects tomorrow's financial constraints.² While we will derive our analytical results under the general case $\beta > 0$, our numerical explorations will be restricted to the case $0 < \beta < 1$, which appears to be realistic from an economic point of view.

Although financial units may be willing to expand their activity (and thus to increase their financial exposure) in the short-run when their expectations are optimistic, we assume that they are reluctant to go beyond a

¹In our notation the time index of the expectation operator refers to a given point in time, while the time index of the dynamic variables C_t , Y_t , K_t^* , etc. refers to a given period of time. Thus for instance $\mathbb{E}_{t-1}[C_t]$ denotes the expectation of the financial ratio in period t taken at the end of period $t-1$ (beginning of period t).

²We have shown elsewhere (see [3]) that β depends in general both on the prevailing interest rate and on the way expectations about future cash inflows/outflows are revised over time.

“threshold” of financial fragility, which we denote by $1 - \mu$ ($0 < \mu < 1$), beyond which the intertemporal financial ratio is considered unsafe. It is thus reasonable to assume that financial units adjust their desired current financial ratio on the basis of the distance between the intertemporal financial ratio K_t^* and its safe level $1 - \mu$. When $K_t^* \geq 1 - \mu$, financial units reduce their desired current expenditures in order to reduce K_t^* ; on the other hand, when $K_t^* < 1 - \mu$ financial units tend to expand the size of their financial outflows. This leads to the following dynamic equation for the desired current financial ratio K_t

$$K_{t+1} = K_t - \alpha[K_t^* - (1 - \mu)]$$

where $\alpha > 0$ is the financial units’ reaction parameter to perceived risk. The “normal” case is characterized by α in the range $(0, 1)$, while $\alpha \geq 1$ describes a situation in which financial units overreact to the fluctuations of financial fragility.

To complete the specification of the model, let us use E^{pu} to denote the volume of public expenditure (assumed to be exogenously given) and E_t to denote the aggregate financial outflows at time t , such that:

$$E_t = E_t^{pr} + E^{pu}$$

Finally, the model is closed by assuming a lag of one period between aggregate financial outflows and production (or aggregate inflows), such that

$$Y_{t+1} = E_t$$

The ranges of the dynamic variables are subject to some restrictions. First, we set a non negativity constraint on the desired current financial ratio K_t . Moreover, it is reasonable to assume that the intertemporal financial ratio cannot exceed an upper bound beyond which most of financial units would go bankrupt. In this paper, as in [2], we assume that this threshold is equal to one at the aggregate level. The difference between one and the value of the aggregate intertemporal financial ratio may be taken as a measure of financial fragility as it expresses the maximum size of a shock that can be beared by the system without changing its qualitative dynamic behavior.

3 The dynamical system

Under the assumed restrictions on K_t and K_t^* , the model derived in the previous section results in the following 4-dimensional nonlinear, *piecewise smooth* dynamical system

$$K_{t+1}^* = \min \left[K_t^* + \beta \left(\frac{E_t - E^{pu}}{Y_t} - K_t^* \right), 1 \right] \quad (1)$$

$$K_{t+1} = \max \{K_t - \alpha[K_t^* - (1 - \mu)], 0\} \quad (2)$$

$$Y_{t+1} = E_t \quad (3)$$

$$E_{t+1} = K_{t+1}Y_t + E^{pu} = \max \{K_t - \alpha[K_t^* - (1 - \mu)], 0\} Y_t + E^{pu} \quad (4)$$

Note that when $K_t - \alpha[K_t^* - (1 - \mu)] < 0$ we get $K_{t+1} = 0$ and $E_{t+1} = E^{pu}$.³ Our purpose is to show, by means of both analytical tools and numerical simulation, that the dynamics specified through the model (1)-(4) is able to generate endogenously fluctuations of the current and intertemporal financial ratios. These fluctuations are qualitatively different, depending on the values of the key parameters α and β . The various possible cases also have different economic interpretations. More precisely, when both adjustment parameters are very low, a globally stable steady state (where financial ratios are at the stationary level $1 - \mu$) is reached by the system in the long-run; when β is low and the reaction parameter α is sufficiently high (but still in the “normal” range, $0 < \alpha < 1$), fluctuations occur along a smooth attracting closed curve, which is reached by the system independently of the initial state. Interestingly enough, however, more complex dynamic outcomes are possible when financial units overreact, i.e. when $\alpha > 1$. In particular, complex behavior may in this case emerge in two different ways, in both of which the ceiling imposed on K_t^* appears to play a crucial role. First of all, for very high α and β a strange attractor exists,⁴ where the dynamics of the system is chaotic; here the role played by the ceiling on K_t^* is that of changing the asymptotic dynamics (i.e. the nature of financial fluctuations) from regular to irregular. Second, for high α and intermediate levels of β a particular regime of the parameters exists where a stable steady state coexists with an attracting closed curve,⁵ in this case the ceiling on K_t^* appears to be responsible for the abrupt appearance of a competing attractor and for the intricate structure of the basins of attraction – which ensures that the long-run outcome of the system is path-dependent.

3.1 Steady states

Let us now determine and characterize the steady states of the model. The equilibrium points of the model are found by imposing $(K_{t+1}^*, K_{t+1},$

³Under the law of motion defined by (1)-(4), non negativity conditions are automatically fulfilled by the other dynamic variables too, provided that one assumes $0 \leq K_0^* \leq 1$, $K_0 \geq 0$, $Y_0 \geq 0$, $E_0 \geq E^{pu}$ and $0 < \beta < 1$. Indeed, in this case we get $Y_t, E_t \geq E^{pu}$; $K_t^* \geq 0$, i.e. we obtain “economically feasible” trajectories.

⁴In Section 4.2 we present an example with $\beta = 0.95$ and α ranging from approximately 2 to 2.5; many similar cases can be found in the range $\beta > 0.5$, $\alpha > 2$. See Section 4.2 for details.

⁵In Section 4.3 we present an example with $\beta = 0.2$ and α ranging approximately from 1.25 to 1.5; many similar cases can be found in the ranges $0.15 < \beta < 0.5$, $1 < \alpha < 2$. See Section 4.3 for details.

$Y_{t+1}, E_{t+1}) = (K_t^*, K_t, Y_t, E_t) = (\bar{K}^*, \bar{K}, \bar{Y}, \bar{E})$ in the system of equations (1)-(4), i.e. by solving:⁶

$$\bar{K}^* = \min \left[\bar{K}^* + \beta \left(\frac{\bar{E} - E^{pu}}{\bar{Y}} - \bar{K}^* \right), 1 \right] \quad (5)$$

$$\bar{K} = \bar{K} - \alpha[\bar{K}^* - (1 - \mu)] \quad (6)$$

$$\bar{Y} = \bar{E} \quad (7)$$

$$\bar{E} = \{ \bar{K} - \alpha[\bar{K}^* - (1 - \mu)] \} \bar{Y} + E^{pu} \quad (8)$$

In order to determine the steady states of the system, notice that from (6) we obtain:

$$\bar{K}^* = 1 - \mu$$

Then, from (7) and (8) we obtain:

$$\bar{Y}(1 - \bar{K}) = E^{pu}$$

Finally, from (5):

$$\bar{K}^* = \min [\bar{K}^* + \beta (\bar{K} - \bar{K}^*), 1]$$

where $\bar{K}^* = (1 - \mu) < 1$, so that:

$$\bar{K} = \bar{K}^* = 1 - \mu$$

and therefore:

$$\bar{Y} = \frac{E^{pu}}{1 - \bar{K}} = \frac{E^{pu}}{\mu}$$

Summarizing, the dynamical system of our model has a unique stationary state, defined by $P \equiv (\bar{K}^*, \bar{K}, \bar{Y}, \bar{E}) \equiv \left(1 - \mu, 1 - \mu, \frac{E^{pu}}{\mu}, \frac{E^{pu}}{\mu} \right)$.⁷

⁶Conditions (6) and (8) are imposed by assuming $\bar{K} > 0$ at the steady state. On the other hand, it can be easily proved that no steady states of the system (1)-(4) exist where $\bar{K} = 0$.

⁷While the equilibrium financial ratios are determined by their exogenous target level $(1 - \mu)$, the equilibrium levels of Y_t and E_t have a straightforward economic interpretation. Indeed, given that the desired current financial ratio $K_t = E_t^{pr} / Y_{t-1}$ can be interpreted as the marginal propensity to financial expenditure of the financial units, the structure of the equation that defines the equilibrium level of Y_t and E_t (i.e. $\bar{Y} = E^{pu} / \mu = E^{pu} / (1 - \bar{K})$), is therefore of the type

$$\frac{\text{autonomous expenditure}}{1 - \text{marginal propensity to financial expenditure}}$$

i.e., a financial multiplier equation similar to the Keynesian multiplier equation.

3.2 Local stability and bifurcations

In this section we discuss the conditions of local asymptotic stability of the unique steady state, which are derived in detail in the Appendix. It can be shown that the steady state is locally asymptotically stable in the region of the space of the parameters α , β , μ which satisfies the following set of inequalities

$$\alpha > 0, 0 < \beta < \frac{2-\mu}{1-\mu}, 0 < \mu < 1 \quad (9)$$

$$\frac{2\mu(\beta-2)}{\beta(2-\mu)} < \alpha < 1 + 3(1-\mu) + \frac{2\mu}{\beta} \quad (10)$$

$$G(\alpha, \beta; \mu) = \beta(1-\mu)^2\alpha^2 + [2\beta^2(1-\mu)^2 - \beta(3\mu^2 - 7\mu + 4) + \mu(2-\mu)]\alpha + \mu(2-\mu)[\beta(1-\mu)(\beta-2) - \mu] < 0 \quad (11)$$

Fig. 1 represents the stability region (shaded area) in the plane of the parameters α , β , for a fixed value of $\mu = 0.25$. As can be verified numerically, the stability region shows very little sensitivity to changes in the fixed parameter μ . Moreover, in the particular case represented in the figure, condition (10) plays a role only for $\beta > 2$, an eventuality that we have excluded.

FIG. 1 APPROXIMATELY HERE

On the curve of equation $G(\alpha, \beta; \mu) = 0$ two eigenvalues of the Jacobian are complex conjugate of modulus equal to unity; a crossing of that curve from inside to outside the shaded area results in a Neimark-Sacker bifurcation where the steady state changes from a stable to unstable focus. The figure clearly shows that this can occur both when β is low and α high enough, and for sufficiently high values of α and β . In other words, if we restrict our analysis to the range $0 < \beta < 1$, the steady state can lose stability when the financial units' reaction parameter (α) is very high and the sensitivity of the intertemporal financial ratio (β) is close to one, but also when sensitivity is low and the financial units' reaction is strong enough.

4 Global dynamics

As already suggested by the local analysis of the previous section, the dynamic scenarios of the model show a remarkable sensitivity with respect to the parameters α and β . We know that the parameter α represents the speed with which the financial units' desired current financial ratio K_t reacts to the deviation of the intertemporal financial ratio K_t^* from its safe level $(1-\mu)$. Thus, high values of α denote overreaction on the part of the financial units. On the other hand the parameter β measures the intensity

with which the current financial ratio affects the future financial constraints and thus the intertemporal financial ratio. This section examines the role played by α and β in the *global* dynamics of the system. The term “global” refers to those dynamic phenomena which cannot be explained by looking only at the linearization of the dynamical system around its equilibrium point. In particular, the aim of the section is twofold: on the one hand, we explore the dynamics of the model for those ranges of the parameters for which the steady state is unstable (parameter combinations outside the stability region of Fig. 1). On the other hand, it is our intention to discuss the *global stability* of the steady state for parameter ranges inside the stability region of Fig. 1. We will limit our analysis to the case $0 < \beta < 1$, which appears to be more realistic from an economic point of view. Within this case moreover, we will distinguish further between the “normal” case, where $0 < \alpha < 1$, and the case of “overreaction”, where $\alpha \geq 1$. The remaining parameters μ and E^{pu} will be set equal to $\mu = 0.25$ and $E^{pu} = 50$ in all our numerical examples, and this implies that the coordinates of the steady state are $(\bar{K}^*, \bar{K}, \bar{Y}, \bar{E}) \equiv (0.75, 0.75, 200, 200)$.

4.1 Endogenous fluctuations without bankruptcy

First of all, we start from sufficiently low levels of the parameters α and β and analyze the dynamic effects of increasing the parameter α . The parameter β is set at the level $\beta = 0.1$. For this level of β , the local stability conditions derived in the Appendix (and represented in Fig. 1) imply that the steady state is locally asymptotically stable (an attracting focus) for low levels of α (as shown in Figs. 2*a,b*), while when α is increased beyond a certain threshold ($\alpha \simeq 0.702$), a Neimark-Sacker bifurcation occurs to the steady state, which becomes a repelling focus. Moreover, numerical simulation provides evidence that the Neimark-Sacker bifurcation is of supercritical type (at least when β is sufficiently low), i.e., for parameter values beyond the Neimark-Sacker boundary a stable closed curve exists (see Figs. 2*c,d* which represent the motion on the closed curve for $\alpha = 0.75$). Then, the effect of further increasing α is that the size of the limit cycle, and therefore the amplitude of the fluctuations, increases (Figs. 2*e,f*). It is remarkable that, when a stable limit cycle exists for low values of β , as in the cases shown in Fig. 2, the range of the fluctuations in the current financial ratio is in general wider than that of the fluctuations in the intertemporal financial ratio. This is because with a low β the intertemporal financial ratio adjusts slowly to changes in the current financial exposure. As a consequence, a characteristic feature of this regime is that financial units never go bankrupt on average, even when α takes very high values, as in the two cases shown in Figs. 2*e,f*.

FIGS. 2*a,b,c,d,e,f* APPROXIMATELY HERE

As suggested by the stability region represented in Fig. 1, a very similar bifurcation scenario (supercritical Neimark-Sacker bifurcation and fluctuations of increasing amplitude) is found also by decreasing β starting from the same initial parameter combination used in Fig. 2a ($\beta = 0.1$, $\alpha = 0.5$).

4.2 A first “route” to complexity: bankruptcy and chaotic dynamics

Now we consider a different scenario, namely one with α and β both high, implying that financial units react strongly to deviation of the intertemporal financial ratio K_t^* from its target level $1 - \mu$ (high α) and that the sensitivity of K_t^* to current financial structure is very strong (high β). More precisely, we set $\beta = 0.95$ and increase α from 2 to 2.5. Unlike the case with a low β , the steady state proves now to be locally stable even for very high values of the reaction parameter α (see again the stability region in Fig. 1), but the (supercritical) Neimark-Sacker bifurcation (which occurs at $\alpha \simeq 2.019$) brings about an oscillatory regime where the desired current financial ratio and the intertemporal financial ratio have fluctuations of similar amplitude (see Figs. 3a,b). The reason for this is that, in the present case, changes in the levels of current financial exposure have significant effects in the long-run. As a consequence, financial units are likely to go bankrupt (Fig. 3c). From a mathematical point of view this causes the ceiling of K_t^* to force the dynamics and to produce chaotic fluctuations where both K_t and K_t^* may vary over time in an unpredictable fashion (see Figs. 3d,e,f). In the “chaos plot” reported in Fig. 4,⁸ each point of the space of the parameters (α, β) is indicated with a different color depending on the value of the largest Lyapunov exponent. The picture confirms the existence of chaotic dynamics in the region of the space of the parameters characterized by high α and high β .

FIGS. 3a,b,c,d,e,f, and FIG. 4 APPROXIMATELY HERE

4.3 A second “route” to complexity: multiple scenarios and path-dependence

In this section we explore a third dynamic scenario, with intermediate values of β , namely with β sufficiently low, but higher than in the example given in

⁸The chaos plot of Fig. 4 is obtained with the software DYNAMICS 2, accompanying Nusse-Yorke’s book [4]. A point of the plane (α, β) is marked with color if the largest Lyapunov exponent (L) is greater than or equal to a predetermined value \bar{L} . The estimate of L is obtained with M pre-iterations and N iterations. We choose $M = N = 1000$, $\bar{L} = 0.01$. The particular color which is used to mark the point depends on the value of L : here we use *black* if $\bar{L} < L < \bar{L} + 0.05$, *dark grey* if $\bar{L} + 0.05 < L < \bar{L} + 0.10$, *light grey* if $\bar{L} + 0.10 < L < \bar{L} + 0.20$.

section 4.1.⁹ What one finds in this parameter regime is that for parameter combinations within the stability region of Fig. 1, but sufficiently close to the Neimark-Sacker boundary, the steady state is stable only locally, *but not globally*, because another attractor exists in the phase-space. In most of our numerical simulations this attractor is a stable closed curve of great amplitude, characterized by periodic or quasi-periodic motion. Fig. 5 follows the sequence of qualitative changes that occur to the attractors and to the associated *basins of attraction* when $\beta = 0.2$ and α is allowed to vary from $\alpha = 1.278$ to $\alpha = 1.45$. In the present section, the basins of attraction of the stable steady state and of the coexisting attractor are represented using different colors. In Fig. 5, for example, the light grey region is the basin of the stable steady state; the dark grey region is the basin of the attracting limit cycle. The black region in the upper left corner, on the other hand, represents the set of initial conditions that generate divergent trajectories, i.e. the so-called *basin of infinity*. Although the phase-space is four-dimensional (and the same holds for the basins of attraction of the two coexisting attractors) we have represented the basins in the two-dimensional (K^*, K) plane, by keeping the initial values E_0 and Y_0 of the state variables E, Y , constant and allowing the initial values of the other state variables (K_0^* and K_0) to vary. In all our numerical examples, we have taken E_0 and Y_0 equal to their common equilibrium value $\bar{Y} = \bar{E} = E^{pu}/\mu = 200$. The initial condition is then represented in the (K_0^*, K_0) -plane with a different color, depending on the long-run behavior of the trajectory generated by the initial condition itself.¹⁰ The starting situation (with $\alpha = 1.278$) is characterized by a stable steady state which is the unique “bounded” attractor. For a higher α , a competing stable closed curve exists in the phase space, whose appearance is due to a *global bifurcation*. In general, the mechanisms of appearance or disappearance of stable (or unstable) closed curves - often associated with *saddle-node connections* - may be quite complex even in the case of two-dimensional discrete-time dynamical systems (see [5] for a computer-assisted study of a wide class of such bifurcations). The bifurcation which has occurred here is a kind of *border-collision* bifurcation, due

⁹More precisely, the dynamic phenomena described in this section can be detected numerically by choosing β approximately in the range $(0.15, 0.5)$ and α in a way such that the pair (α, β) is inside the stability region and close to the Neimark-Sacker boundary. One can immediately check from Fig. 1 that in this range of β the crossing of the Neimark-Sacker curve is possible only under the assumption of financial units’ overreaction ($\alpha > 1$).

¹⁰Of course, in general one would get a different picture under a different choice of E_0 and Y_0 , because in this case the basins would be computed by taking the initial condition along a different *section* of the high-dimensional phase-space. Furthermore, while the basins of attraction are represented in a given section (a plane) of the high-dimensional phase-space, the high-dimensional attractors are represented by taking their *projections* on the same plane where the initial conditions are taken. Thus (see e.g. Figs. 5c,d) the attractor may seem to “cross” its basin boundary, but of course this is only due to our way of representing in the same plane both the attractors and the basins of a 4-dimensional system.

to the constraint $K^* = 1$, which seems very similar to the one described in [6]. What is remarkable is that the new attractor appears abruptly, and its basin of attraction becomes wider and wider as α increases, while on the other hand the basin of the steady state is gradually reduced to a small neighborhood (Figs. 5*b,c,d,e*).¹¹ In particular, in the case of Fig. 5*e* the stationary point is stable only under very small perturbations around it, while perturbations beyond a certain size may determine the crossing of the basin boundary, such that the system enters a new regime of wide fluctuations.¹² The steady state changes from stable to unstable focus when α has reached its Neimark-Sacker bifurcation value. Numerically, this seems to occur precisely when the basin boundary shrinks around the steady state itself (at $\alpha \simeq 1.444$), i.e. in this case the Neimark-Sacker bifurcation appears to be of *subcritical* type. As one can argue from the examples discussed in the present section and in sections 4.1 and 4.2, the crossing of the Neimark-Sacker boundary in the (α, β) parameter plane may give rise to different types (supercritical or subcritical) of Neimark-Sacker bifurcations. A comparison between the foregoing examples suggests, quite surprisingly, that the Neimark-Sacker boundary contains two *Chenciner points* (instead of one, as it generally occurs), where the nature of the bifurcation changes from supercritical to subcritical, or vice-versa¹³ (see e.g. [10] for a textbook treatment). Indeed, the Neimark-Sacker bifurcation is supercritical for $\beta = 0.1$ and $\alpha \simeq 0.702$ (section 4.1), subcritical for $\beta = 0.2$ and $\alpha \simeq 1.444$ (in the present section), supercritical again for $\beta = 0.95$ and $\alpha \simeq 2.019$ (section 4.2).

Coming back to Fig. 5, note the peculiar “parameter dependence” of the sequence illustrated from Fig. 5*b* to 5*e*: changes of the parameter α within the range (1.279, 1.44) determine a drastic modification of the structure of the basins of attraction, without any important modification of the structure of the attractors.

FIG. 5*a,b,c,d,e,f* APPROXIMATELY HERE

Choosing different values of β , we may find slightly different bifurcation sequences. For instance, the steady state may undergo a supercritical Neimark-Sacker bifurcation (when its basin of attraction is still a wide region of the phase space) such that two different stable closed orbits share the

¹¹Note that a further border-collision bifurcation, which now involves the other constraint $K = 0$, occurs at some intermediate parameter value between those used in Fig. 5*d* and Fig. 5*e*. Again, the mechanism is probably similar to the one described in [6].

¹²In the case of two-dimensional systems this phenomenon is sometimes called corridor stability, following the terminology introduced by Leijonhufvud [7] and Howitt [8]. For a recent application, see e.g. Kind [9].

¹³This may be associated with the fact that the true bifurcation surface is three-dimensional (in the $3 - D$ parameter space) and the *Chenciner set* is a curve which is crossed twice under the section considered in Fig. 1.

state space (see Fig. 6a); in other cases the stable steady state may coexist with a strange attractor (see Fig. 6b).¹⁴

What is remarkable is that such kinds of “global” bifurcation bring about situations of multiple asymptotic states, and path-dependent long-run behavior. In other words, Fig. 5 demonstrates that there exist particular parameter regimes, under which the system will settle down on the stationary state, or fluctuate in the long-run, depending on the initial position. As an example consider Fig. 7 (same parameters as Fig. 5c) where it is shown that slightly different initial conditions determine completely different outcomes in the long-run: given $K_0^* = 0.9$, for $K_0 = 1.7$ the system converges to the steady state, for $K_0 = 1.75$ to the closed orbit, for $K_0 = 1.85$ to the steady state again. This example shows that in general the structure of the basins can be rather irregular: some initial states do not converge to the equilibrium point, though they are close to the steady state; on the other hand initial states which are far from equilibrium give rise to trajectories that converge to the steady state. This is a “kind” of complex behavior associated with the structure of the basins of attraction, rather than with the structure of the attractors. It is obvious that such kind of complexity makes it difficult to predict the effect of an exogenous shock on the long-run dynamics of the system. This raises a serious problem for economic policy aiming to stabilize financial fluctuations (see [2]).

FIGS. 6a,b and FIGS. 7a,b,c APPROXIMATELY HERE

5 Conclusions

The model of financial fluctuations developed in the present paper builds on a stylized representation of the feedback between current and intertemporal financial conditions of the economy. Such an interaction is governed a) by financial units’ speed of adjustment towards desired financial conditions (captured by the parameter α of our model) and b) by the intensity with which tomorrow’s financial conditions are affected by current financial decisions (expressed by the parameter β). Despite its simplicity, the model is capable of producing a wide range of dynamic scenarios according to the values of the parameters. Among them, we have focused on the following

- (i) convergence to a (globally) stable steady state;
- (ii) “regular” fluctuations on an attracting closed curve;
- (iii) chaotic fluctuations;

¹⁴The existence of chaotic dynamics for particular parameter ranges within the region of local asymptotic stability of the steady state is suggested by a comparison of the stability region in Fig. 1 with the chaos plot in Fig. 4.

(iv) path-dependence, when a locally stable steady state coexists with a competing attractor characterized by regular, or chaotic, oscillatory motion.

While scenarios (i) and (ii) are typically detected in the case of weak financial units' reaction (low α), scenarios (iii) and (iv) are possible under financial units' overreaction ($\alpha > 1$), provided that the level of intertemporal financial fragility is sensitive enough to current aggregate financial decisions (sufficiently high β).

Our results suggest that the simple feedback mechanism of this model is likely to capture a key determinant of financial fluctuations.

Appendix

Local stability conditions of the steady state

In this Appendix we derive the conditions of local asymptotic stability of the steady state $P \equiv (\bar{K}^*, \bar{K}, \bar{Y}, \bar{E}) = (1 - \mu, 1 - \mu, E^{pu}/\mu, E^{pu}/\mu)$. Notice first that, despite the ceiling imposed on K_t^* and the non negativity constraint set on K_t , the map associated with the dynamical system (1)-(4) is differentiable at the steady state. Indeed, in a neighborhood of the steady state, equations (1), (2) and (4) become $K_{t+1}^* = K_t^* + \beta[(E_t - E^{pu})/Y_t - K_t^*]$, $K_{t+1} = K_t - \alpha[K_t^* - (1 - \mu)]$ and $E_{t+1} = \{K_t - \alpha[K_t^* - (1 - \mu)]\} Y_t + E^{pu}$, respectively. The Jacobian matrix of the system evaluated at the equilibrium point is the following

$$\mathbf{J} = \begin{bmatrix} 1 - \beta & 0 & -\beta\mu(1 - \mu)/E^{pu} & \beta\mu/E^{pu} \\ -\alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha E^{pu}/\mu & E^{pu}/\mu & 1 - \mu & 0 \end{bmatrix}$$

while the characteristic equation is given by

$$z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0 \quad (12)$$

where:

$$\begin{aligned} a_1 &= \beta - 2 \\ a_2 &= \mu - \beta + \alpha\beta = \mu - (1 - \alpha)\beta \\ a_3 &= (1 - \mu)(2 - \beta - \alpha\beta) = (1 - \mu)[2 - (1 + \alpha)\beta] \\ a_4 &= (1 - \mu)(\beta - 1) \end{aligned}$$

A necessary and sufficient condition for (12) to only have roots of ab-

solute value less than one is the following (see [11] p. 399):

$$a_4 < 1 \quad (13)$$

$$3 + 3a_4 > a_2 \quad (14)$$

$$1 + a_1 + a_2 + a_3 + a_4 > 0 \quad (15)$$

$$1 - a_1 + a_2 - a_3 + a_4 > 0 \quad (16)$$

$$(1 - a_4)(1 - a_4^2) - a_2(1 - a_4)^2 + (a_1 - a_3)(a_3 - a_1a_4) > 0 \quad (17)$$

We now rewrite the above general conditions in terms of the parameters α, β, μ , by assuming in general $\alpha, \beta > 0, 0 < \mu < 1$. From (13) we get

$$(\beta - 1)(1 - \mu) < 1$$

i.e.

$$\beta < \frac{2 - \mu}{1 - \mu} \quad (18)$$

which is satisfied, in particular, when $0 < \beta < 1$. From (14):

$$3 + 3(\beta - 1)(1 - \mu) > \mu - (1 - \alpha)\beta$$

i.e.

$$\alpha < 1 + 3(1 - \mu) + \frac{2\mu}{\beta} \quad (19)$$

which is satisfied, in particular, in the “normal” case $0 < \alpha < 1$. From (15)

$$1 + \beta - 2 + \mu - (1 - \alpha)\beta + (1 - \mu)[2 - (1 + \alpha)\beta] + (1 - \mu)(\beta - 1) > 0$$

one easily gets

$$\mu\alpha\beta > 0 \quad (20)$$

which is always true for positive values of the parameters.

Condition (16) becomes

$$1 - (\beta - 2) + \mu - (1 - \alpha)\beta - (1 - \mu)[2 - (1 + \alpha)\beta] + (1 - \mu)(\beta - 1) > 0$$

i.e.

$$\alpha\beta(2 - \mu) + 2\mu(2 - \beta) > 0 \quad (21)$$

which is always true for $\beta \leq 2$, while for $\beta > 2$ is satisfied for

$$\alpha > \frac{2\mu(\beta - 2)}{\beta(2 - \mu)}$$

Finally, let us consider (17), i.e.

$$\begin{aligned} & [1 + (1 - \beta)(1 - \mu)] \left[1 - (1 - \beta)^2(1 - \mu)^2 \right] \\ & \quad - (\mu - \beta + \alpha\beta) [1 + (1 - \beta)(1 - \mu)]^2 \\ & \quad + [\alpha\beta(1 - \mu) - (2 - \beta)(2 - \mu)] [\beta(2 - \beta)(1 - \mu) - \alpha\beta(1 - \mu)] > 0 \end{aligned}$$

which can be rewritten as

$$A\alpha^2 + B\alpha + C < 0 \quad (22)$$

where

$$\begin{aligned} A &= \beta^2(1 - \mu)^2 \\ B &= \beta[2\beta^2(1 - \mu)^2 - \beta(3\mu^2 - 7\mu + 4) + \mu(2 - \mu)] \\ C &= \beta\mu(2 - \mu)[\beta(1 - \mu)(\beta - 2) - \mu] \end{aligned}$$

Notice that the discriminant of the left-hand side of (22) can be rewritten as

$$\Delta = B^2 - 4AC = \beta^2[(2 - \mu) - \beta(1 - \mu)]^2[4\beta^2(1 - \mu)^2 - 4\beta\mu(1 - \mu)^2 + \mu^2]$$

where

$$4\beta^2(1 - \mu)^2 - 4\beta\mu(1 - \mu)^2 + \mu^2 = [2\beta(1 - \mu) - \mu]^2 + 4\beta\mu^2(1 - \mu) > 0$$

so that $\Delta \geq 0$ and (22) is reduced to

$$\frac{-B - \sqrt{\Delta}}{2A} < \alpha < \frac{-B + \sqrt{\Delta}}{2A} \quad (23)$$

Moreover, the smaller root $\frac{-B - \sqrt{\Delta}}{2A}$ is negative¹⁵ at least for $\beta \leq \frac{2 - \mu}{1 - \mu}$, and therefore in this case condition (23) can be simplified to

$$0 < \alpha < \frac{-B + \sqrt{\Delta}}{2A}$$

¹⁵The statement follows from the fact that $A > 0$ and that $-B - \sqrt{\Delta} < -B - \beta|2\beta(1 - \mu) - \mu|(2 - \mu) - \beta(1 - \mu)|$, where the right-hand side can be written as follows

$$\begin{aligned} & -B - \beta|2\beta(1 - \mu) - \mu|(2 - \mu) - \beta(1 - \mu)| \\ & = \begin{cases} -2\beta[2\beta^2(1 - \mu)^2 - 2\beta(1 - \mu)(2 - \mu) + \mu(2 - \mu)] \\ \quad \text{(for } 0 < \beta \leq \mu/[2(1 - \mu)] \text{ or } \beta > (2 - \mu)/(1 - \mu)) \\ -2\beta^2\mu(1 - \mu) \\ \quad \text{(for } \mu/[2(1 - \mu)] < \beta \leq (2 - \mu)/(1 - \mu)) \end{cases} \end{aligned}$$

which is negative at least in the range $0 < \beta \leq \frac{2 - \mu}{1 - \mu}$.

Summarizing, the steady state is locally asymptotically stable in the region of the positive orthant of the space of the parameters α , β , μ which satisfies the following set of inequalities

$$\beta < \frac{2 - \mu}{1 - \mu}$$

$$\frac{2\mu(\beta - 2)}{\beta(2 - \mu)} < \alpha < 1 + 3(1 - \mu) + \frac{2\mu}{\beta}$$

$$\beta(1 - \mu)^2\alpha^2 + [2\beta^2(1 - \mu)^2 - \beta(3\mu^2 - 7\mu + 4) + \mu(2 - \mu)]\alpha + \mu(2 - \mu)[\beta(1 - \mu)(\beta - 2) - \mu] < 0$$

which is represented in the plane of the parameters α , β , for a fixed value of $\mu = 0.25$, in Fig. 1.

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FIGURE CAPTIONS

Fig. 1. Region of local asymptotic stability in the plane of the parameters α, β ($\mu = 0.25$)

Fig. 2. Qualitative effects of increasing the reaction parameter α , starting with low α and β ($\mu = 0.25, E^{pu} = 50, \beta = 0.1$)

Fig. 3. Qualitative effects of increasing the reaction parameter α , starting with high α and β ($\mu = 0.25, E^{pu} = 50, \beta = 0.95$)

Fig. 4. Chaos plot for α and β in the range $(0, 2.5)$ and $\mu = 0.25$

Fig. 5. Multiple attractors in the stable regime ($\mu = 0.25, E^{pu} = 50, \beta = 0.2$). Basins of attraction are obtained with $Y_0 = E_0 = E^{pu}/\mu = 200$, K_0^* and K_0 varying in the (K^*, K) -plane

Fig. 6. Other cases of multiple attractors ($\mu = 0.25, E^{pu} = 50$). Basins are obtained with $Y_0 = E_0 = 200$, K_0^* and K_0 varying in the (K^*, K) -plane

Fig. 7. Path dependence in the stable regime ($\mu = 0.25, E^{pu} = 50, K_0^* = 0.9, E_0 = Y_0 = 200$)

Fig. 1

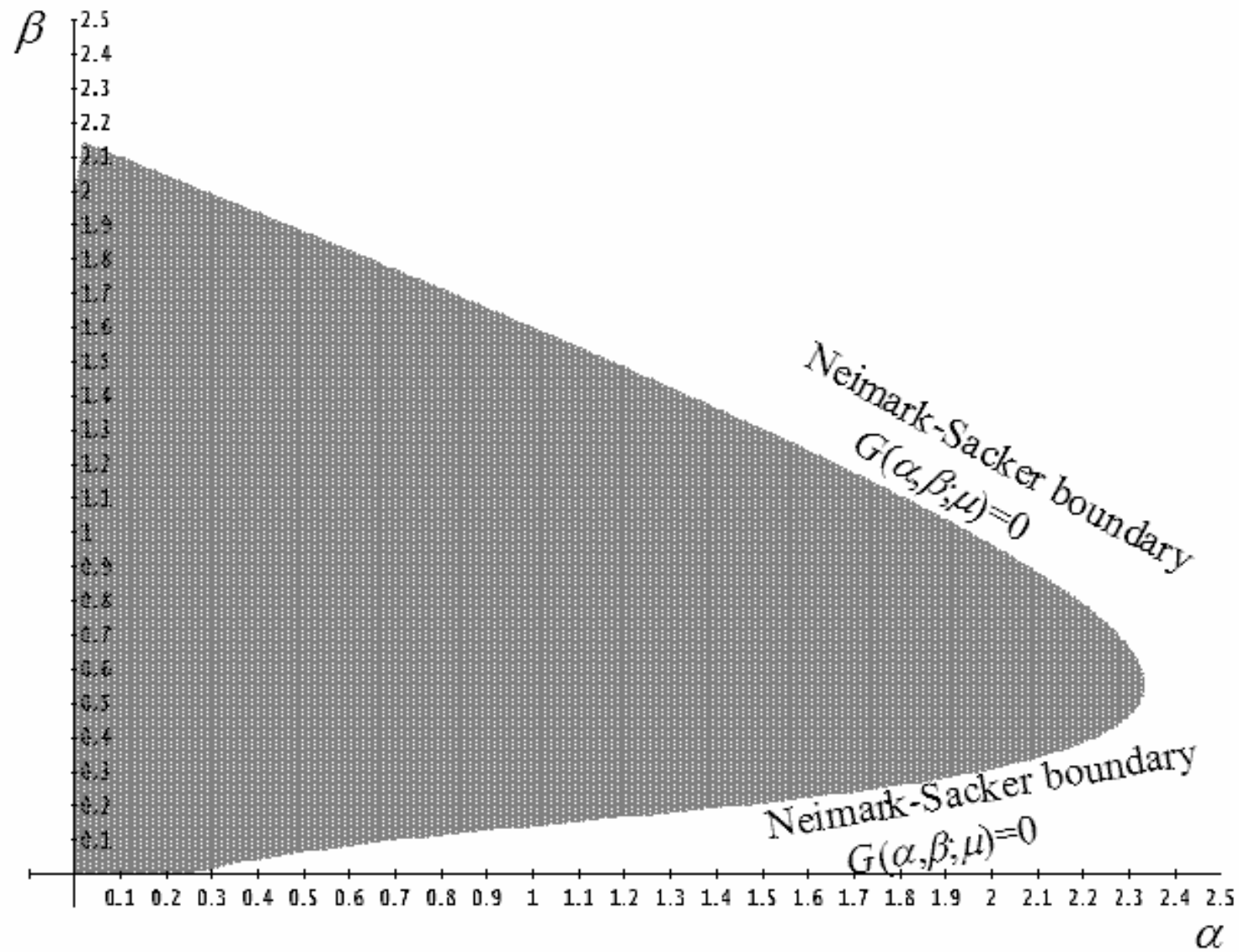


Fig. 2

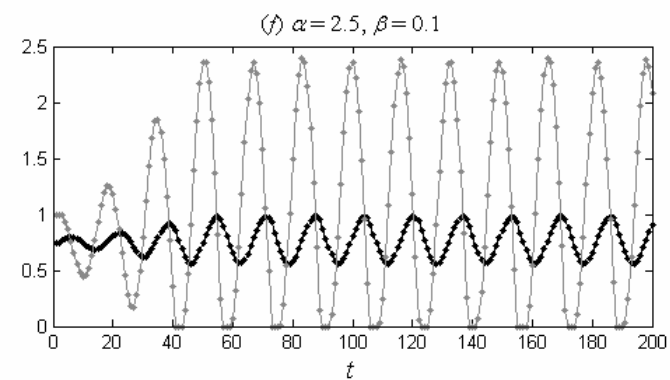
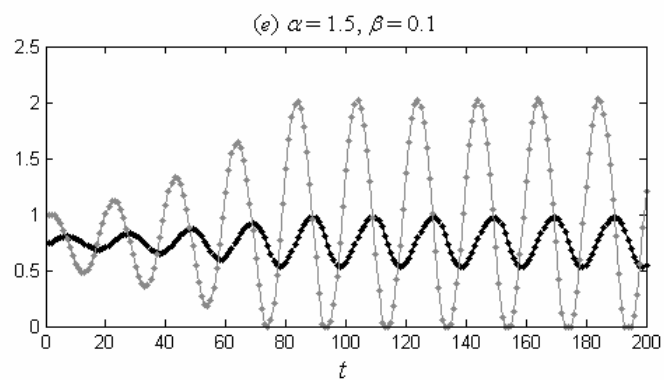
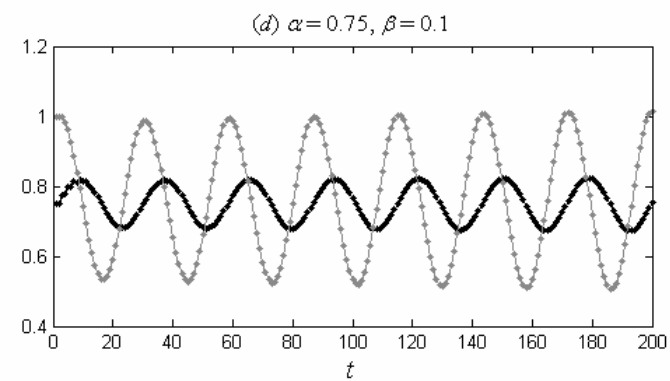
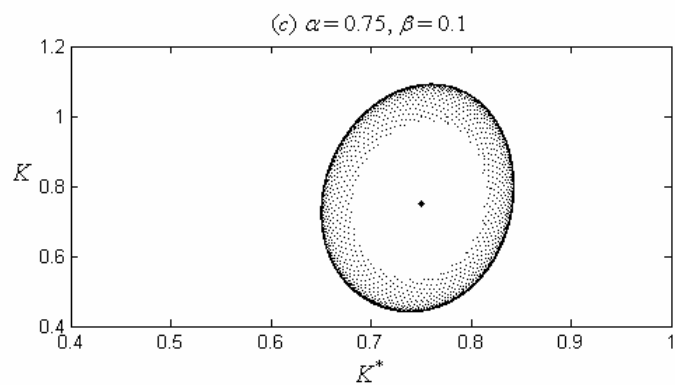
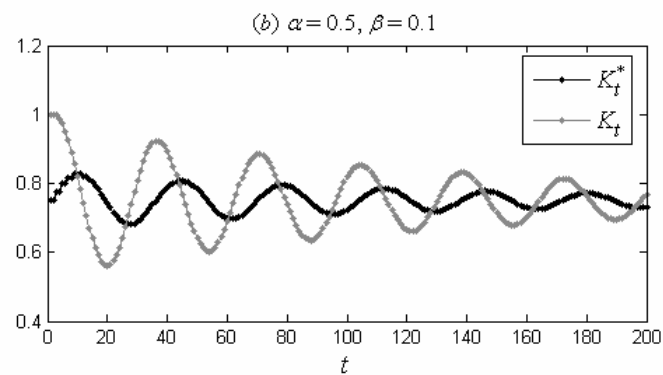
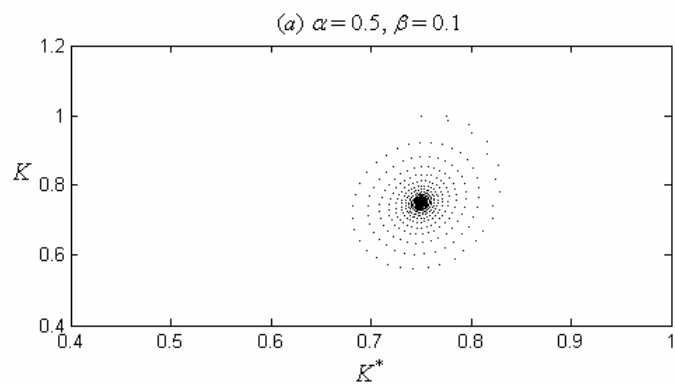


Fig. 3

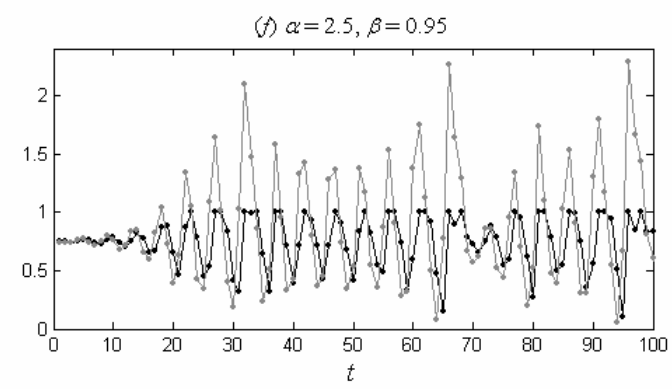
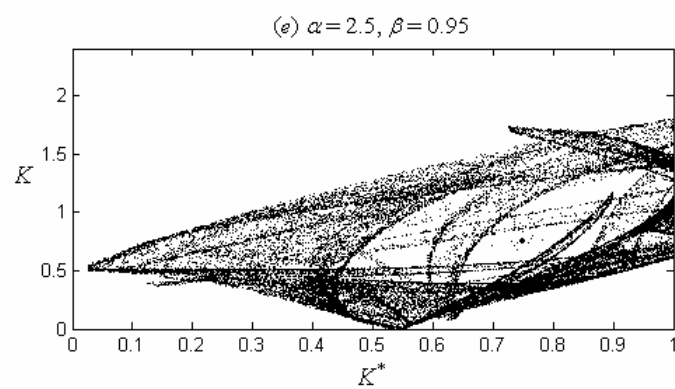
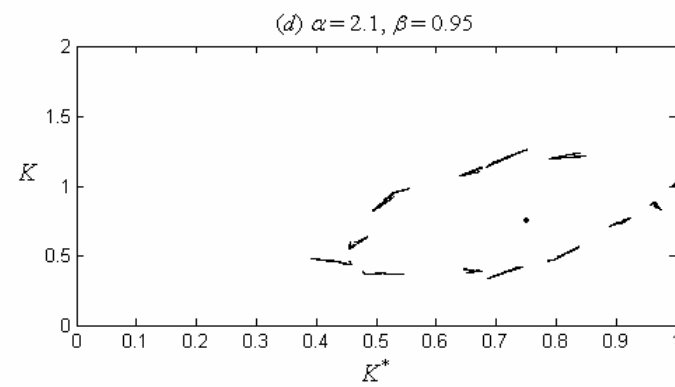
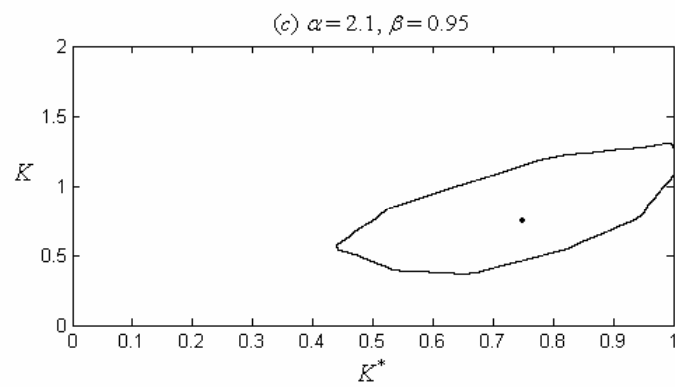
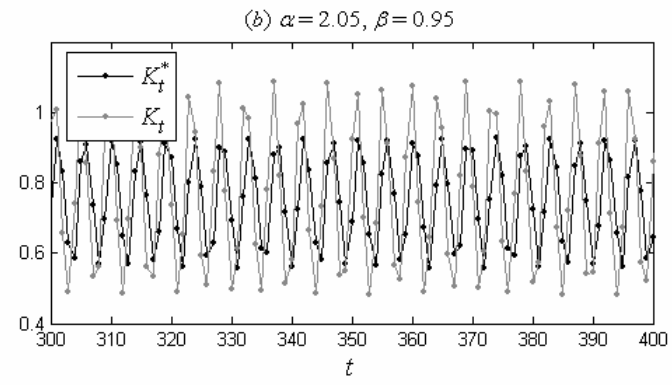
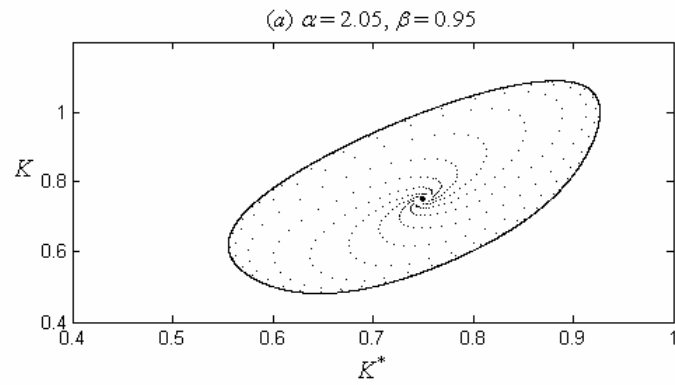


Fig. 4

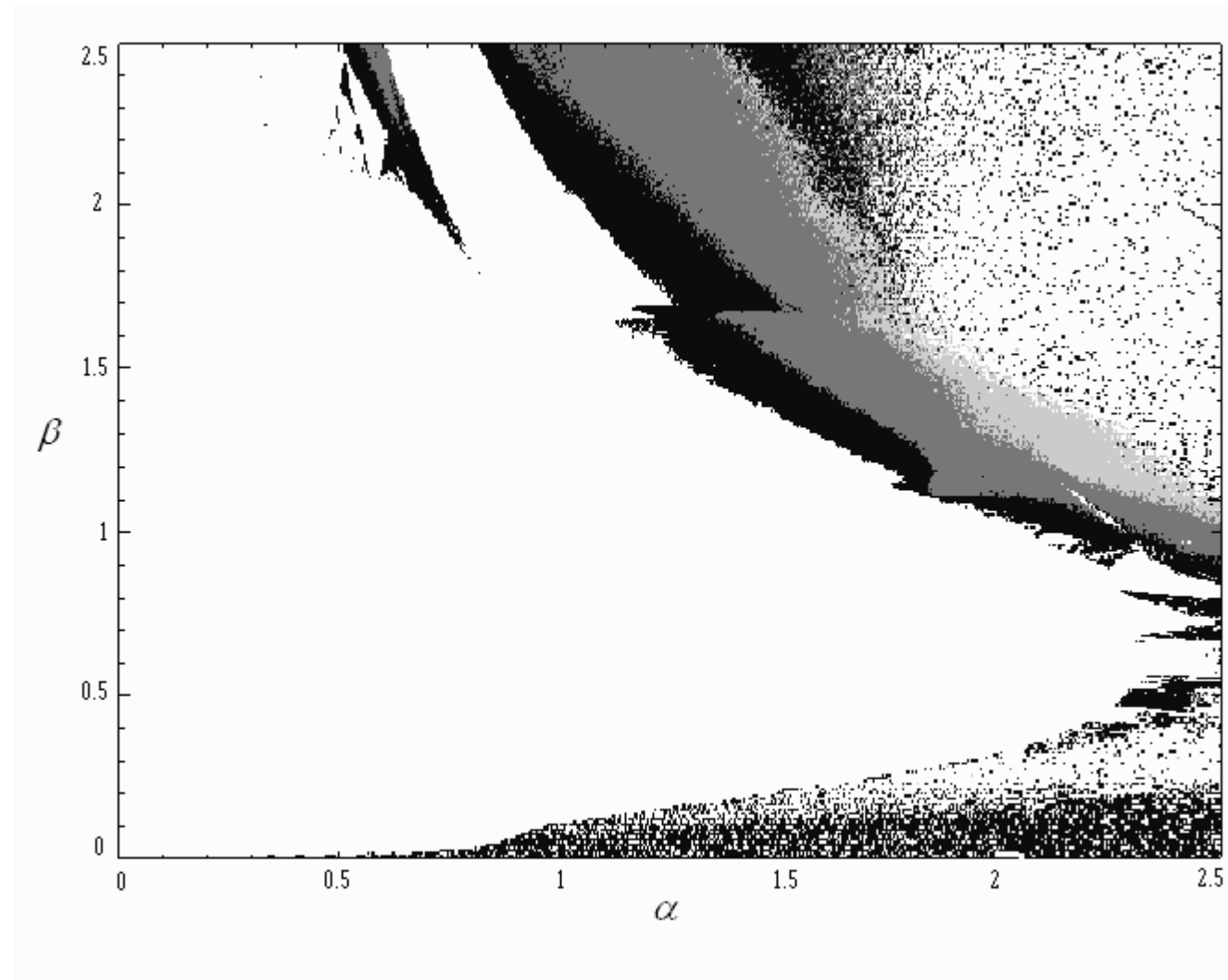


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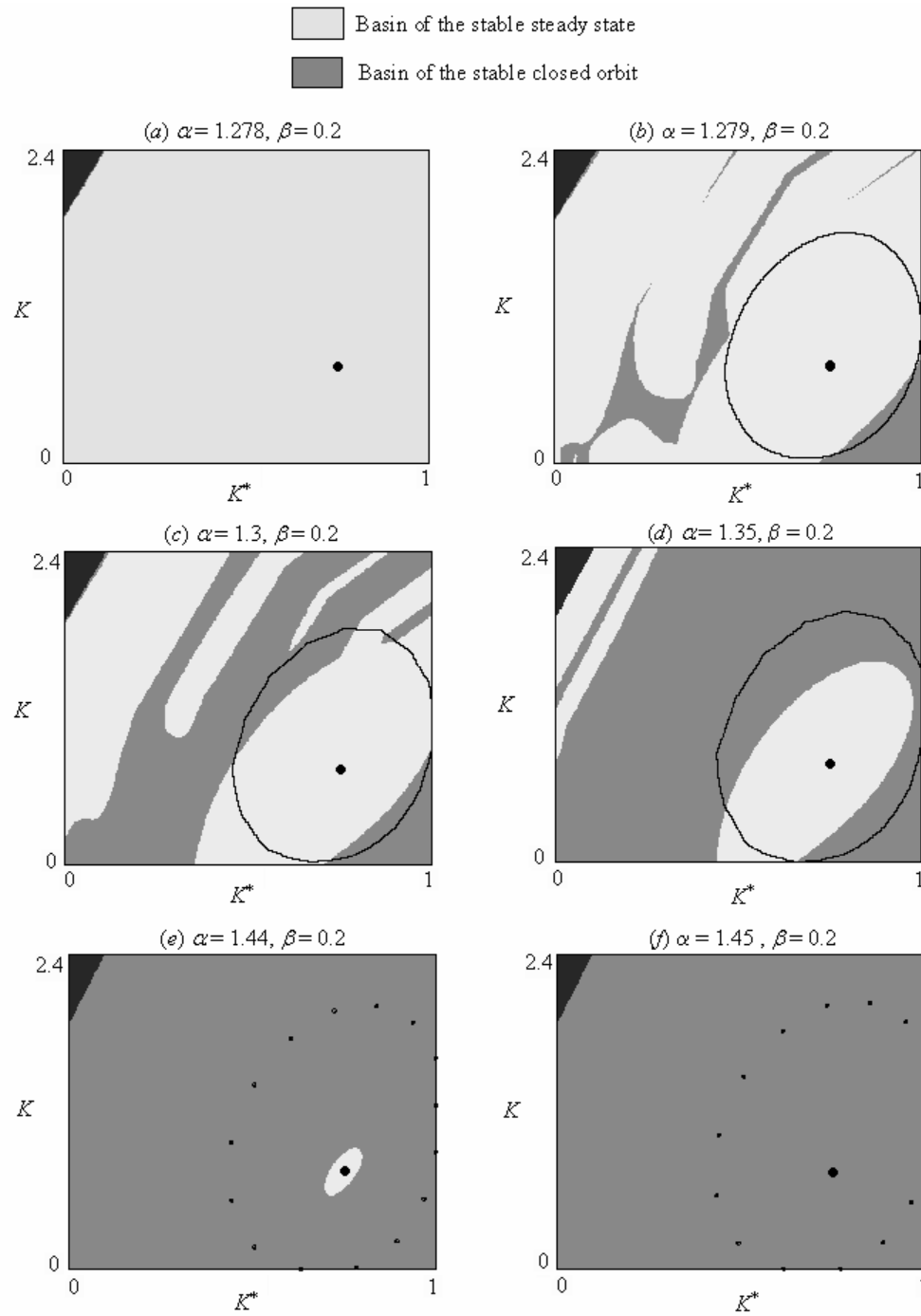


Fig. 6

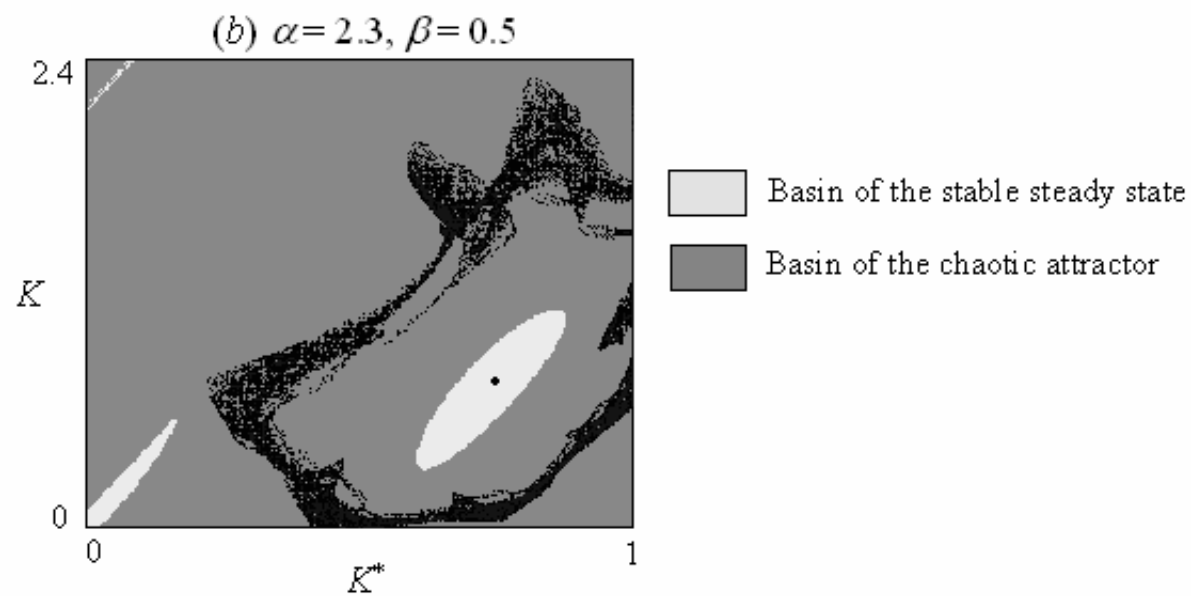
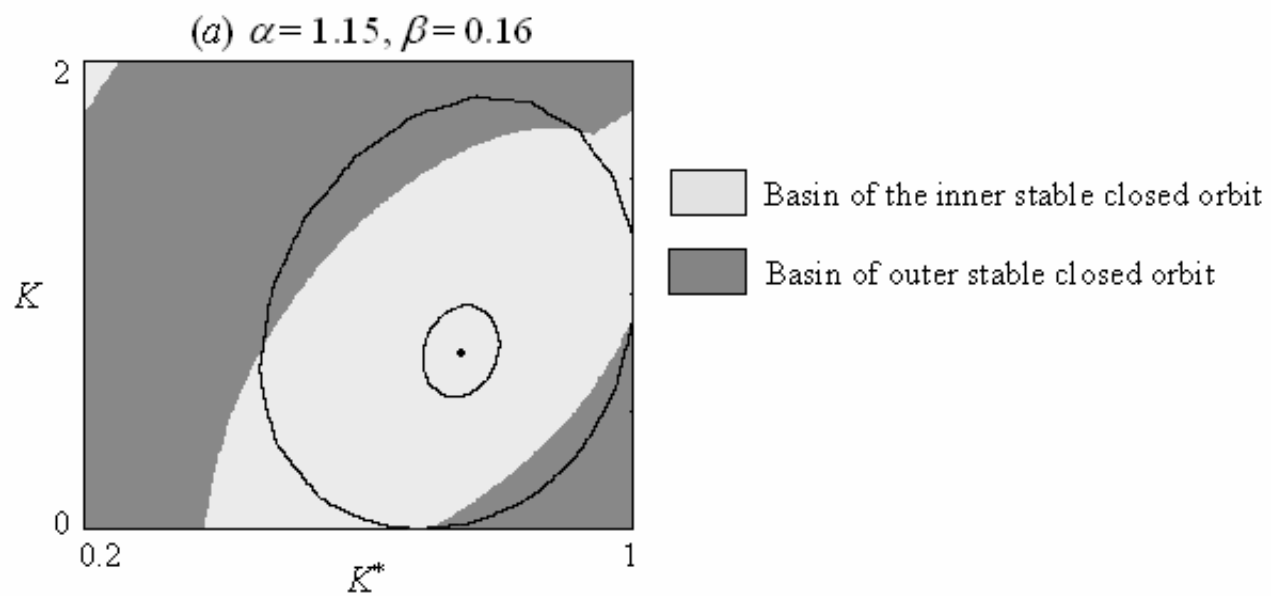


Fig. 7

