Asset price and wealth dynamics in a financial market with heterogeneous agents

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Abstract

This paper considers a discrete-time model of a financial market with one risky asset and one risk-free asset, where the asset price and wealth dynamics are determined by the interaction of two groups of agents, fundamentalists and chartists. In each period each group allocates its wealth between the risky asset and the safe asset according to myopic expected utility maximization, but the two groups have heterogeneous beliefs about the price change over the next period: the chartists are trend extrapolators, while the fundamentalists expect that the price will return to the fundamental. We assume that investors' optimal demand for the risky asset depends on wealth, as a result of CRRA utility. A market maker is assumed to adjust the market price at the end of each trading period, based on excess demand and on changes of the underlying reference price. The model results in a nonlinear discrete-time dynamical system, with growing price and wealth processes, but it is reduced to a stationary system in terms of asset returns and wealth shares of the two groups. It is shown that the long-run market dynamics are highly dependent on the parameters which characterize agents' behavior as well as on the initial condition. Moreover, for wide ranges of the parameters a (locally) stable fundamental steady state coexists with a stable "non-fundamental" steady state, or with a stable closed orbit, where only chartists survive in the long-run: such cases require the numerical and graphical investigation of the basins of attraction. Other dynamic scenarios include periodic orbits and more complex attractors, where in general both types of agents survive in the long run, with time varying wealth fractions.

Keywords: heterogeneous agents, financial market dynamics, wealth dynamics, coexisting attractors

JEL-classification: C61, D84, G12

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1 Introduction

In recent years several models of asset price dynamics based on the interaction of *heterogeneous agents* have been proposed (Day and Huang (1990), Kirman (1991), Brock and LeBaron (1996), Brock and Hommes (1998), Lux (1998), Gaunersdorfer (2000), Chiarella and He (2001a, 2003), Farmer and Joshi (2002), Chen and Yeh (2002), Hommes et al. (2005), to quote only a few). This heterogeneous agent literature can be classified as either theoretically or computationally oriented in general, and has been extensively discussed in two recent surveys by Hommes (2006) and LeBaron (2006). In particular, Hommes' survey discusses the state of the art of analytically tractable heterogeneous agent models[2], whereas LeBaron's survey discusses extensively related work on computational heterogeneous agent based models in finance. The common setup of several heterogeneous agent models in finance is characterized by a stylized market with one risky asset and one riskless asset, and the main focus is on the effect of heterogeneous beliefs and trading rules on the dynamics of the price of the risky security. Most models however, some of which allow the size of the different groups of agents to vary according to the relative profitability of the adopted trading rules, are of necessity rather difficult to treat analytically. Chiarella *et al* (2002) have developed a two-dimensional discrete time model of asset price dynamics that contains the essential elements of the heterogeneous agents paradigm whilst still remaining mathematically tractable. In that paper, a financial market with a risky asset and an alternative asset has been assumed, consisting of two types of traders, *fundamentalists* and *chartists*, and of a *market maker*, who adjusts the price in each period depending on excess demand. In Chiarella *et al* (2002), as well as in most studies on the interaction of heterogeneous agents, the evolution of agents' wealth and its effect on price dynamics is left in the background; indeed, in those papers the underlying assumptions about agents' portfolio allocation follow the framework of Brock and Hommes (1998), where optimal demand for the risky asset is independent on agents' wealth, as a result of the assumption of Constant Absolute Risk Aversion (CARA) utility functions.

In general these assumptions are unrealistic[3]: a more realistic framework, where investors' optimal decisions depend on their wealth, has been proposed and analyzed through numerical simulation by Levy *et al* (1994, 1995). This framework is consistent with the assumption of Constant Relative Risk Aversion (CRRA) utility functions. More recently, Chiarella and He (2001b, 2002) have proved analytically the existence of multiple steady states in financial market models with heterogeneous agents and CRRA utility: in these papers, the main focus is on the existence and stability of multiple equilibria as a function of the trend traders' extrapolation rate.

The present paper aims to contribute to the development and analysis of such models, by analyzing the dynamics of asset price and agents' wealth within a

[2] Some of which are closely related to the one considered in the present paper.

[3] See e.g. Levy et al. (2000) and Campbell and Viceira (2002), for a discussion of this point.

fundamentalists/chartists framework similar to the one developed in Chiarella *et al.* (2002). In addition, we allow for a growing dividend process and a trend in the fundamental price of the risky asset. As a consequence, the model that we develop results in price and wealth being determined simultaneously over time, as in real markets, which gives rise to interdependent growing wealth and price processes[4]. In order to obtain analytical results about the dynamics of the growing system, and to discuss the role played by the key parameters and by the initial conditions, the nonstationary model is reformulated in terms of returns and wealth shares and reduced to a stationary system. Though in the framework of the present paper agents are not allowed to switch among different groups according to the relative profitability of the adopted trading rules, the time-varying wealth shares modify the impact the two groups on the market demand, thus determining endogenously time varying weights of fundamentalists and chartists. As a consequence, differently from the fundamentalists-chartists CARA framework depeloped in Chiarella *et al.* (2002), we are now able to characterize the equilibria and the other kinds of asymptotic behaviour in terms of long-run evolution of the wealth proportions.

The structure of the paper is as follows. Section 2 presents the general framework of the model. In particular Section 2.1 derives the asset demand of a generic agent, as a function of his/her beliefs about the risky return, in a framework consistent with CRRA utility. Section 2.2 derives a benchmark notion of fundamental solution, which plays a role in fundamentalist expectations formation. Section 2.3 describes the schemes used by fundamentalists and chartists to form and revise their expectations, and thus provides a specification of the demand function of each group. Section 2.4 describes how demands are aggregated by a market maker, who sets the price depending on excess demand. Section 3 presents the resulting nonlinear dynamical system for the dynamic evolution of fundamental value and price, agents' expected returns, and wealth of the two groups. In Section 3.1 this is reduced to a stationary map in terms of actual and expected capital gain of the risky asset, fundamental to price ratio, and wealth shares of the two groups. The steady states are determined and their properties are discussed in Section 3.2. Numerical exploration of the global behavior and discussion of the main dynamic scenarios, as well as stochastic simulations are contained in Section 4. Section 5 concludes. Mathematical details are provided in the Appendices.

2 The model

We consider a discrete-time model of a financial market with one risky asset and one riskless asset, two types of interacting agents, *fundamentalists* and *chartists* (denoted by $j \in \{f, c\}$), and a *market maker*. The starting point is Chiarella *et al.* (2002), whose antecedents are Chiarella (1992), Beja and Goldman (1980),

[4] Related papers that investigate non-stationary heterogeneous agents models with growing dividends given by a geometric random walk, are Brock and Hommes (1997b) and Hommes (2002).

and Zeeman (1974).

Each group has CRRA utility of wealth function. We denote, at time t, by $\Omega_t^{(j)}$ the wealth of group j and by $Z_t^{(j)}$ the fraction of wealth that agent-type j invests in the risky asset. We also denote by P_t and P_t^* the market price and the *fundamental* price of the risky asset, respectively, by r ($r > 0$) the (constant) risk-free rate, by D_t the (random) dividend, while D_t/P_{t-1} is the dividend yield in period t. The fundamental 'reference' price P_t^*, which will be defined in Section 2.2, is assumed to be known to the fundamentalists and to the market maker.

As we shall see in the next section, the fraction $Z_t^{(j)}$ is independent of the wealth level under our assumptions, and therefore the demand $\Omega_t^{(j)} Z_t^{(j)}$ for the risky asset is proportional to the wealth level $\Omega_t^{(j)}$. The number of agents within each group is assumed fixed. Given that traders of the same group are assumed to have the same beliefs, risk aversion and trading strategies, the proportion of wealth invested in the risky asset at each point in time will be the same for all the agents of the same group. This implies that the distribution of wealth among the agents of the same type has no influence on the dynamics, but only total wealth for each group matters. As a consequence, we can normalise to one the number of agents of each group, and use indifferently the terms agent j or agent-type j (or group j). Wealth of agent-type j evolves according to

$$\begin{aligned} \Omega_{t+1}^{(j)} &= \Omega_t^{(j)} + \Omega_t^{(j)} Z_t^{(j)} \left(\frac{P_{t+1} + D_{t+1} - P_t}{P_t} \right) + \Omega_t^{(j)} (1 - Z_t^{(j)}) r = \\ &= \Omega_t^{(j)} \left[1 + r + Z_t^{(j)} \left(\frac{P_{t+1} + D_{t+1} - (1+r) P_t}{P_t} \right) \right] \qquad (1) \end{aligned}$$

where $(P_{t+1} + D_{t+1} - (1+r)P_t)/P_t$ represents the excess return in period $t+1$.

2.1 Asset demand

Each agent is assumed to have a CRRA (power) utility of wealth function of the type

$$u^{(j)}(\Omega) = \begin{cases} \frac{1}{1-\lambda^{(j)}} (\Omega^{1-\lambda^{(j)}} - 1) & (\lambda^{(j)} \neq 1) \\ \ln(\Omega) & (\lambda^{(j)} = 1) \end{cases}$$

where $\Omega > 0$ and the parameter $\lambda^{(j)} > 0$ represents the relative risk aversion coefficient.

Denote by $E_t^{(j)}$, $Var_t^{(j)}$ the "beliefs" of agent-type j about expectation and variance. Each agent seeks the investment fraction $Z_t^{(j)}$ maximizing the expected utility of wealth at time $t+1$, that is $\max_{Z_t^{(j)}} E_t^{(j)}[u^{(j)}(\Omega_{t+1}^{(j)})]$. As is well known, an analytical exact solution to this problem can be obtained only under very particular assumptions[5]. Chiarella and He (2001b) derive the following approximate optimal solution (which we adopt in the present paper)

[5] See e.g. Campbell and Viceira (2002).

$$Z_t^{(j)} = \frac{E_t^{(j)}[(P_{t+1}+D_{t+1}-P_t)/P_t - r]}{\lambda^{(j)} Var_t^{(j)}[(P_{t+1}+D_{t+1}-P_t)/P_t - r]} = \frac{E_t^{(j)}[\rho_{t+1}+\delta_{t+1}-r]}{\lambda^{(j)} Var_t^{(j)}[\rho_{t+1}+\delta_{t+1}-r]},$$

where $\rho_{t+1} \equiv (P_{t+1} - P_t)/P_t$ and $\delta_{t+1} \equiv D_{t+1}/P_t$ denote the *capital gain* and the *dividend yield*, respectively. Therefore, $Z_t^{(j)}$ is proportional to agent j's "risk-adjusted" expected excess return.

2.2 Fundamental price

Following the framework of Brock and Hommes (1998), Appendix 1 derives endogenously a reference notion of "fundamental solution", as a long-run market clearing price path which would be obtained under homogeneous beliefs about expected excess return. Furthermore this price is assumed to satisfy a long-run stability condition, namely the "no-bubbles" condition.

Here we focus on the particular case of zero net supply of shares[6], where the fundamental price can be derived from the "no-arbitrage" equation

$$E_t[P_{t+1} + D_{t+1}] = (1+r)P_t \ . \tag{2}$$

As it is well known, the unique solution to the expectational equation (2) which satisfies the no-bubbles transversality condition, $\lim_{k\to+\infty} E_t[P_{t+k}]/(1+r)^k = 0$, is given by

$$P_t = P_t^* \equiv \sum_{k=1}^{\infty} \frac{E_t[D_{t+k}]}{(1+r)^k}. \tag{3}$$

In particular, in the case of an i.i.d. dividend process $\{D_t\}$ with $E_t[D_{t+k}] = \overline{D}$, $k = 1, 2, ...$, the *fundamental* solution (3) is constant, given by $P_t^* = P^* = \overline{D}/r$, while in the case of a dividend process described by[7]

$$E_t[D_{t+k}] = (1+\phi)^k D_t, \quad k = 1, 2, ..., \tag{4}$$

with $0 \le \phi < r$, the fundamental solution is given by

$$P_t^* = (1+\phi)D_t/(r-\phi). \tag{5}$$

We will use the latter specification of the dividend process: as one can easily check this implies that the fundamental evolves over time according to[8]

$$E_t[P_{t+1}^*] = (1+\phi)P_t^*, \tag{6}$$

and that along the fundamental path the expected dividend yield and the capital gain are given respectively by

$$E_t[\delta_{t+1}] \equiv E_t\left[\frac{D_{t+1}}{P_t^*}\right] = r - \phi, \qquad E_t[\rho_{t+1}] \equiv E_t\left[\frac{P_{t+1}^* - P_t^*}{P_t^*}\right] = \phi \ .$$

[6] In doing so we follow Brock and Hommes (1998) and Chiarella and He (2001a). Appendix 1 discusses the general case of positive net supply.

[7] This case is known in the finance literature as "Gordon growth model"

[8] Which reduces to $E_t[P_{t+1}^*] = P_t^*$ in the particular case $\phi = 0$.

In the next section we introduce heterogeneity into the model, by taking the view that agents form heterogeneous, time-varying beliefs about the first and second moment of returns, though they are assumed to share the same beliefs about expected dividends (according to (4)).

2.3 Heterogeneous beliefs

The two groups differ in the way they form and update their beliefs about the price change over the next period. We introduce heterogeneous beliefs both about the expectation and the variance of the excess return.

The *fundamentalists* believe that the price will return to the (known) fundamental in the future, so that their expected price change is given by

$$\begin{aligned} E_t^{(f)}[P_{t+1} - P_t] &= E_t^{(f)}[P_{t+1}^* - P_t^*] + \eta(P_t^* - P_t) = \\ &= \phi P_t^* + \eta(P_t^* - P_t). \end{aligned}$$

The fundamentalist rule is based on the expected change in the underlying fundamental and includes a correction term, proportional to the difference between fundamental and current price, which depends on their beliefs about the speed of mean reversion (captured by the parameter η, $0 < \eta < 1$). We also assume that fundamentalist conditional variance is constant over time, $Var_t^{(f)}[\rho_{t+1} + \delta_{t+1}] = \sigma_f^2$. The fundamentalist investment fraction in the risky asset thus becomes

$$Z_t^{(f)} = \frac{1}{P_t} \frac{\eta(P_t^* - P_t) + \phi P_t^* + (1+\phi)D_t - rP_t}{\lambda^{(f)}\sigma_f^2}. \tag{7}$$

The *chartists* do not rely on the knowledge of the fundamental price. Instead, they behave as *trend followers* and try to extrapolate into the future the movements of past prices. Therefore, the chartists' conditional expected price change evolves over time according to a weighted average (with geometrically declining weights) of past capital gains, which results in the adaptive rule

$$\begin{aligned} m_t^{(c)} &\equiv E_t^{(c)}\left[\rho_{t+1}\right] = E_t^{(c)}\left[\frac{P_{t+1} - P_t}{P_t}\right] = \\ &= (1-c)m_{t-1}^{(c)} + c\left(\frac{P_t - P_{t-1}}{P_{t-1}}\right) = (1-c)m_{t-1}^{(c)} + c\rho_t, \end{aligned}$$

where the parameter c, $0 < c < 1$, represents the weight given to the most recent price change.

We introduce a risk-adjustment mechanism into the demand function of the chartists, by assuming that they have time-varying beliefs about the variance of the excess return. Precisely, we assume that their beliefs about the second moment are *state-dependent*, in the sense that chartists expect that market risk increases when the expected excess return moves away (above or below) from "normal" levels, e.g. during phases of booms and crashes. Put differently, in our model the chartists behave as speculators who try to exploit the current price trends. However, they perceive the risk that a market characterized for instance by very high expected returns might collapse. This leads to our assumption of state-dependent beliefs about risk. A simple way to model this idea (similar to Chiarella *et al* (2002)), is to assume that at each time step the estimated variance $\sigma^2_{c,t} \equiv Var^{(c)}_t[\rho_{t+1}+\delta_{t+1}-r]$ depends positively on the *magnitude* of the expected excess return, i.e. $\sigma^2_{c,t} = v(x_t)$, $x_t \equiv E^{(c)}_t[\rho_{t+1}+\delta_{t+1}-r]$, with $v'(x) > 0$ ($v'(x) < 0$) for $x > 0$ ($x < 0$), and where $v(0) > 0$ represents a minimum level of variance corresponding to the "normal" case of zero expected excess return. Note that under this simplifying assumption the chartist investment fraction in the risky asset turns out to depend only on the expected excess return, i.e. $Z^{(c)}_t = Z^{(c)}(x_t) = x_t/(\lambda^{(c)}v(x_t))$. In spite of this risk-adjustment mechanism, we assume however that chartists are more sensitive to the chance of higher returns than to the related risk, and that variations of $v(x)$ are not large enough as to compromise the monotonic shape of the function $Z^{(c)}(x)$: namely, we assume that the function $v(x)$ is "inelastic", $|v'(x)| < v(x)/|x|$.[9] In particular, we adopt a function $Z^{(c)}(x)$ with an increasing and bounded S-shaped graph, whose slope levels off as the magnitude $|x_t|$ of the expected excess return increases. We use the following two-parameter specification

$$Z^{(c)}_t = \frac{\gamma}{\theta}\tanh\{\theta x_t\} = \frac{\gamma}{\theta}\tanh\left\{\theta[m^{(c)}_t + (1+\phi)D_t/P_t - r]\right\}, \tag{8}$$

where γ, $\theta > 0$, which provides a sufficiently flexible formulation. Indeed the parameter $\gamma = 1/(\lambda^{(c)}v(0))$[10], which is inversely related to chartists' risk aversion, represents the slope of the chartist investment fraction $Z^{(c)}(x)$ computed for $x = 0$ (see *Fig. 1a*), that we will call *strength* of chartist demand; moreover, for a given γ, the parameter θ governs the position of the floor and the ceiling for the fraction of wealth invested in the risky asset (see *Fig. 1b*). *Fig. 1c* represents the "implied" variance function $v(x)$ corresponding to the case with $\theta = 50$ and $v(0) = 0.0025$.

***** FIG. 1 approximately here *****

[9] Note that $xv'(x) < v(x)$ is equivalent to $dZ^{(c)}/dx > 0$. Moreover, our assumptions about $v(x)$ imply also $dZ^{(c)}/dx < Z^{(c)}(x)/x$. Note also that the assumed qualitative properties do not prevent $v(x)$ from being convex, as is the case of the function used in our model. Of course other convex functions - for instance the simple quadratic function $v(x) = v(0) + ax^2$, $a > 0$ - may lead to non-monotonic demand functions. We are grateful to one of the referees for having raised the question of the connection between the qualitative behavior of $v(x)$ and that of $Z^{(c)}(x)$.

[10] Our choice of a hyperbolic tangent to represent the S-shaped chartist investment in the risky asset implies an estimated variance given by $v(x) = v(0)\theta x/\tanh\{\theta x\}$, where $v(0) \equiv \lim_{x\to 0} v(x)$, whose graph is represented in *Fig. 1c*. In this case the function $v(x)$ is strictly convex and approaches asymptotically the straight lines of equation $f(x) = \mp v(0)\theta x$ as $x \to \infty$. Of course other specifications are possible, for example an unbounded S-shaped function could also be consistent with our assumptions about the variance $v(x)$. Analytical and numerical study of the model with alternative specifications show that what really matters in order to get the key dynamic features of the model are the assumed qualitative properties of $v(x)$ and $Z^{(c)}(x)$.

We note that because of equation (5) we obtain $(1+\phi)D_t = (r-\phi)P_t^*$, so that the demand functions (7) and (8) can be rewritten, respectively as

$$Z_t^{(f)} = \frac{(\eta+r)(P_t^* - P_t)/P_t}{\lambda^{(f)}\sigma_f^2}, \quad Z_t^{(c)} = \frac{\gamma}{\theta}\tanh\left\{\theta[m_t^{(c)} + (r-\phi)P_t^*/P_t - r]\right\}.$$

2.4 Price setting rule

Price adjustments are operated by a *market maker*, who is assumed to know the fundamental reference price[11]. The market maker clears the market at the end of period t by taking an off-setting long or short position and announces the next period price as a function of agents' *excess demand* and expected changes in the reference price. In the general case, the assumed price setting rule of the market maker will be given by

$$P_{t+1} - P_t = E_t^{(m)}[P_{t+1}^* - P_t^*] + P_t\ H_t(N_t^D - N_t^S), \tag{9}$$

where $E_t^{(m)}[P_{t+1}^* - P_t^*] = E_t^{(f)}[P_{t+1}^* - P_t^*] = \phi P_t^*$ is the market maker's expected change in the underlying fundamental, $N_t^D = (\Omega_t^{(f)} Z_t^{(f)} + \Omega_t^{(c)} Z_t^{(c)})/P_t$ is the total agents' demand at time t (number of shares), N_t^S denotes the supply of shares at time t, and it is assumed in general that $H_t(\cdot)$ is a strictly increasing function, with $H_t(0) = 0$. In eq. (9) the term $P_t H_t(N_t^D - N_t^S)$ represents the price change due to excess demand, while $E_t^{(m)}[P_{t+1}^* - P_t^*]$ captures the price adjustment due to news about the underlying fundamental price. The price setting rule (9) is a very stylized one, which however can be justified in the light of recent theoretical and empirical work on financial market microstructure (see e.g. the survey in Madhavan (2000)): for instance, Madhavan and Smidt (1993) develop a market maker model of a financial market with informed and uninformed traders, where the specialist seeks to maximize his/her final wealth by choosing both the price quote and the trade. It is found that under the optimal specialist's policy, the change in the price quote has three components: *(i)* a term depending on order imbalances, *(ii)* a term which captures market maker's revision of beliefs about the underlying asset value and *(iii)* a term depending on the desired change in inventories. In our stylized framework, in order to keep the model simple and to focus on the behaviour of heterogeneous traders, we only consider the impact of excess demand and changes in the underlying asset value; the role of inventories is left in the background, though of course this represents an important aspect of the market maker decision problem in real markets.

Notice that total agents' demand N_t^D (number of shares) can be rewritten as $N_t^D = Z_t\Omega_t/P_t$, where $\Omega_t = \Omega_t^{(f)} + \Omega_t^{(c)}$ is the total wealth and $Z_t \equiv (\Omega_t^{(f)} Z_t^{(f)} + \Omega_t^{(c)} Z_t^{(c)})/\Omega_t$ is the fraction of total wealth invested in the risky asset at time t. Denoting by $Q_t \equiv N_t^S P_t/\Omega_t$ the value of the supply of shares as a fraction

[11] The use of the market maker mechanism to clear the market in fundamentalist-chartist models goes back at least to Beja and Goldman (1980) and Day and Huang (1990).

of total agents' wealth, we also obtain $N_t^S = Q_t\Omega_t/P_t$, so that $N_t^D - N_t^S = (Z_t - Q_t)\Omega_t/P_t$. We model the market maker rule as independent of the level of growing prices and wealth (so that it is not affected by the level of Ω_t/P_t) i.e. we assume

$$H_t(N_t^D - N_t^S) = H(Z_t - Q_t),$$

where H is strictly increasing with $H(0) = 0$. We will adopt the linear specification $H(Z_t - Q_t) = \beta(Z_t - Q_t)$, $\beta > 0$. In particular, in the case of zero net supply we get $H(Z_t - Q_t) = H(Z_t) = \beta Z_t$.

3 The dynamical system

Under the assumed stochastic processes of the dividends and fundamental price, the dynamics of the model will be given in general by a "noisy" nonlinear dynamical system. In this paper we focus on the dynamics of the "deterministic skeleton" of the model, i.e. we assume that dividends evolve in a deterministic way according to their commonly shared expected value (Section 4.1). Some numerical simulations of a stochastic version of the model are contained in Section 4.2. The deterministic dynamics can be summarized as:

$$\begin{aligned}
P_{t+1} &= P_t + \phi P_t^* + P_t\beta Z_t, && (10)\\
m_{t+1}^{(c)} &= (1-c)m_t^{(c)} + c[(P_{t+1} - P_t)/P_t], && (11)\\
P_{t+1}^* &= (1+\phi)P_t^*, && (12)\\
\Omega_{t+1}^{(j)} &= \Omega_t^{(j)}\left[1 + r + Z_t^{(j)}\left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t}\right)\right],\ j \in \{f, c\} && (13)
\end{aligned}$$

where:

$$\begin{aligned}
\Omega_t &= \Omega_t^{(f)} + \Omega_t^{(c)}, \quad Z_t = (\Omega_t^{(f)}Z_t^{(f)} + \Omega_t^{(c)}Z_t^{(c)})/\Omega_t\ ,\\
Z_t^{(f)} &= \frac{(\eta + r)(P_t^* - P_t)/P_t}{\lambda^{(f)}\sigma_f^2}, \quad Z_t^{(c)} = \frac{\gamma}{\theta}\tanh\left\{\theta[m_t^{(c)} + (r-\phi)P_t^*/P_t - r]\right\}.
\end{aligned}$$

Although the system results in general in growing price[12] and wealth processes, it is possible to obtain a stationary[13] system in terms of *capital gain* $\rho_{t+1} \equiv (P_{t+1} - P_t)/P_t$, *fundamental/price ratio* $y_t \equiv P_t^*/P_t$, and *wealth shares* of fundamentalists and chartists $w_t^{(j)} \equiv \Omega_t^{(j)}/\Omega_t$, $j \in \{f, c\}$, with $w_t^{(c)} = (1 - w_t^{(f)})$.

Moreover, we denote by

$$g_{t+1}^{(j)} = r + Z_t^{(j)}\left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t}\right), \qquad j \in \{f, c\},$$

[12] Note however that the underlying deterministic model, in the particular case $\phi = 0$, reduces to the case of constant fundamental.

[13] The term stationary here is used in the sense that we reduce the growing system to a form which admits steady state solutions.

the growth rate of wealth of agent-type j, and by

$$g_{t+1} = r + Z_t \left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t} \right) = w_t^{(f)} g_{t+1}^{(f)} + (1 - w_t^{(f)}) g_{t+1}^{(c)}$$

the rate of growth of total wealth. Notice also that (in the deterministic skeleton of the model) the actual excess return on the risky asset in period $t+1$ can be rewritten as

$$\begin{aligned} \frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t} &= \frac{P_{t+1} - P_t}{P_t} + \frac{(r-\phi)P_t^*}{P_t} - r = \\ &= \rho_{t+1} + (r-\phi)y_t - r. \end{aligned}$$

In particular, a dynamic equation for the wealth shares can be obtained by rewriting the wealth recurrence equations (1) for $j \in \{f, c\}$ as

$$w_{t+1}^{(j)} \Omega_{t+1} = w_t^{(j)} \Omega_t (1 + g_{t+1}^{(j)}), \tag{14}$$

By summing up equations (14) for $j \in \{f, c\}$, and recalling that $w_{t+1}^{(f)} + w_{t+1}^{(c)} = 1$ we obtain

$$\Omega_{t+1} = \Omega_t \left[w_t^{(f)} (1 + g_{t+1}^{(f)}) + w_t^{(c)} (1 + g_{t+1}^{(c)}) \right] = \Omega_t (1 + g_{t+1}),$$

and finally eq. (14) becomes for $j = f$

$$w_{t+1}^{(f)} = w_t^{(f)} (1 + g_{t+1}^{(f)}) \frac{\Omega_t}{\Omega_{t+1}} = \frac{w_t^{(f)} (1 + g_{t+1}^{(f)})}{(1 + g_{t+1})}.$$

3.1 The map

The changes of variables performed in the previous section allow us to rewrite the original dynamical model in terms of the state variables ρ_t, y_t, $m_t^{(c)}$, $w_t^{(f)}$. The time evolution of the stationary system is given by the iteration of the following non linear map $T : (\rho, y, m^{(c)}, w^{(f)}) \longmapsto (\rho', y', m^{(c)\prime}, w^{(f)\prime})$, where the symbol $'$ denotes the unit time advancement operator:

$$T : \begin{cases} \rho' = \phi y + \beta Z, \\ y' = y(1+\phi)/(1+\rho'), \\ m^{(c)\prime} = (1-c)m^{(c)} + c\rho', \\ w^{(f)\prime} = w^{(f)}[1 + r + Z^{(f)}(\rho' + (r-\phi)y - r)]/[1 + r + Z(\rho' + (r-\phi)y - r)], \end{cases} \tag{15}$$

where

$$\begin{aligned} Z^{(f)} &= \frac{(\eta + r)(y-1)}{\lambda^{(f)} \sigma_f^2}, \quad Z^{(c)} = \frac{\gamma}{\theta} \tanh[\theta(m^{(c)} + (r-\phi)y - r)], \\ Z &= w^{(f)} Z^{(f)} + (1 - w^{(f)}) Z^{(c)}. \end{aligned}$$

Although in (15) we have 4 dynamic variables, the map T could in fact be immediately rewritten as a $3-D$ map, given that ρ' is a function of y, $m^{(c)}$, and $w^{(f)}$. However, we keep the higher dimensional form (15), in order to consider explicitly the behavior of the dynamic variable ρ (capital gain)

3.2 Steady states

We now discuss the existence and stability of the steady states of the map T. As proven in the Appendix 2, the map (15) has two types of steady states, that we denote by *fundamental steady states* and by *non-fundamental steady states*, respectively. The map also presents other important *trapping*[14] subsets of the phase-space.

Fundamental steady states. Fundamental steady states are characterized by

$$\begin{gathered} y = 1, \\ \rho = m^{(c)} = \phi, \\ w^{(f)} = \overline{w}^{(f)}, \quad \overline{w}^{(f)} \in [0,1], \end{gathered}$$

i.e. by the price being at the fundamental ($P_t = P_t^*$, for any t) and growing at the fundamental rate ϕ, and by zero excess demand, $Z = 0$. The long-run wealth distribution $(\overline{w}^{(f)}, 1-\overline{w}^{(f)})$ at a fundamental steady state may be given, in general, by any $\overline{w}^{(f)} \in [0,1]$. This means that a *continuum* of steady states exists[15]: numerical simulations of the dynamical system in Section 4.1 show that the steady state wealth distribution which is reached by the system in the long-run depends on the initial condition.

Non-fundamental steady states. For particular ranges of the parameters other steady states exist, coexisting with the fundamental ones. They are characterized by

$$\begin{gathered} y = 0, \\ w^{(f)} = 0, \\ \rho = m^{(c)} = \overline{\rho}, \end{gathered}$$

where $\overline{\rho}$ solves

$$\overline{\rho}/\beta = (\gamma/\theta)\tanh[\theta(\overline{\rho} - r)]. \tag{16}$$

It can be shown (see Appendix 2) that the non-fundamental growth rates $\overline{\rho}$ which come out as positive solutions of (16) are higher than the risk free rate r (and thus higher than the fundamental growth rate ϕ). Nonfundamental steady states are thus characterized by the price growing faster than the fundamental, $\rho = m^{(c)} = \overline{\rho} > r > \phi$, the fundamental/price ratio converging to 0, $y = 0$, i.e. $\lim_{t\to\infty} P_t/P_t^* = +\infty$, the market dominated by chartists, $w^{(f)} = 0$, and a permanent positive excess demand $Z = Z^{(c)} = (\gamma/\theta)\tanh[\theta(\overline{\rho} - r)] > 0$. Numerical simulations show that, for wide ranges of the parameters, a *locally attracting* non-fundamental steady state exists. Of course, the existence of such attracting non-fundamental equilibrium, where the price increasingly deviates from the fundamental and the fundamentalist wealth becomes negligible, represents a situation which cannot be sustained in the long-run: this is a sort of "deterministic bubble" which lasts forever. This dynamic outcome is due, first

[14] A subset X of the phase space is trapping if it is mapped into itself, i.e. $T(X) \subseteq X$.

[15] They are located in a one-dimensional subset (a straight line) of the phase-space.

of all, to the deterministic nature of the model being considered here, which represents the "skeleton" of a stochastic dynamic model. Once noise is added to the system, the behavior of price and wealth becomes more realistic and often phases of booms alternate with phases of crashes in an unpredictable way, with a complex evolution of prices and wealth shares (see Section 4.2). Second, the present model is a partial equilibrium model, where the permanent excess demand of over-optimistic chartists causes the asset price to grow faster than its fundamental, even in the case of zero outside supply and zero "fundamental" risk premium: the reason is that the excess demand is assumed to be always satisfied by the market maker within the present model, without any constraint from the side of inventories. A further reason may be that in our simple framework agents' consumption, and its effect on wealth dynamics, is not taken into account.[16] Nevertheless, the existence and stability of such equilibria provides the important information that bubbles may start to develop for particular initial conditions and under particular sets of parameters.

Trapping sets. Further important trapping subsets of the phase-space represent the cases where only fundamentalists ($w^{(f)} = 1$) or only chartists ($w^{(f)} = 0$) populate the market.

In the case $w^{(f)} = 1$ (the market dominated by the fundamentalists) the evolution of the model is determined by the following map

$$T^{(f)} : \begin{cases} \rho' = \phi y + \beta Z^{(f)}, \\ y' = y(1+\phi)/(1+\rho'), \end{cases}$$

where the dynamic equation for $m^{(c)}$ has been neglected given that it has no influence on the dynamics of ρ and y. One can prove very easily (see Appendix 2) that in this case the fundamental steady state is the only equilibrium characterized by non negative growth rate of the price. Moreover, the model in this case can be rewritten as a one-dimensional system, and the fundamental steady state is locally stable for sufficiently low expected speed of mean reversion (η), high risk aversion ($\lambda^{(f)}$), low speed of market reaction (β).

In the case $w^{(f)} = 0$ (the market dominated by the chartists) the dynamics are driven by the map[17]

$$T^{(c)} : \begin{cases} \rho' = \phi y + \beta Z^{(c)}, \\ y' = y(1+\phi)/(1+\rho'), \\ m^{(c)\prime} = (1-c)m^{(c)} + c\rho'. \end{cases}$$

In this particular case it can be proven that (see again Appendix 2):

- the "fundamental" equilibrium $y = 1$, $\rho = m^{(c)} = \phi$, is locally asymptotically stable for low values of c (strength of extrapolation), β (price

[16] This is indeed an important issue, as recently stresses in a different framework by Hens and Schenk-Hoppé (2005), who argue that the way consumption is modelled - or not modelled at all - affects the long-run evolution of the wealth shares of agents with heterogeneous trading strategies.

[17] Which could be rewritten as a two-dimensional map

reaction) and γ (i.e. for high chartists' risk aversion), as can be seen from the stability domain represented in *Fig. 2*;

- for higher values of c, β and γ the fundamental steady state is no longer attracting. As shown in *Fig. 2*, when $\Delta \equiv \beta\gamma$ or c are varied so that the bifurcation curve of equation $\Delta = \Delta_{NS}(c) = (c+\phi)/[c(1+r)-(r-\phi)]$ is crossed from inside the stability region, then the steady state becomes unstable via a Neimark-Sacker bifurcation[18], which is followed by the appearance of a *stable* closed curve. For parameter selections far from the Neimark-Sacker boundary, the system converges to an attracting limit cycle (with long-run fluctuations of the price around the fundamental) or to a non-fundamental equilibrium (with permanent and increasing deviation of the price away from the fundamental).

Numerical simulations show that the attractors of the map $T^{(c)}$ (limit cycle, non-fundamental steady state) are in general attractors also for the map T, i.e. they can be reached starting with $w_0^{(f)} > 0$ as well. This fact has two implications. From the mathematical point of view, this enhances the importance of the analysis of the particular case $w_0^{(f)} = 0$, which proves very useful in understanding the dynamics of the system in the general case. From the evolutionary point of view, this means that for a given set of initial conditions (with nonzero wealth shares of the two groups) the trend followers not only are able to survive in the long-run, but also tend to accumulate more wealth than the fundamentalists.

***** FIG. 2 approximately here *****

4 Numerical simulation of the global dynamics

This section contains numerical experiments aimed at gaining some insight into the global dynamics of the model. Overall, these simulations show the range of dynamic scenarios that the dynamic model is able to generate and, more importantly, they provide evidence that the fundamentalists, in general, do not accumulate more wealth than the chartists. In particular the analysis will focus on the situations of coexistence of fundamental and nonfundamental steady states (or coexistence with other attractors), with an analysis of the role of initial conditions and basins of attraction. Some numerical simulations of a stochastic version of the model are also provided.

4.1 Deterministic dynamics

All the phase-space representations throughout the present section are obtained by means of projections in the plane of the variables ρ, y, except for the basins of attraction in *Fig. 5* (where the initial conditions are taken in the plane $w^{(f)}, y$).

[18] We have numerical evidence of the supercritical nature of the Neimark-Sacker bifurcation, with a stable limit cycle issuing from the bifurcated steady state.

Fig. 3 is devoted to the dynamic behavior of the model when the market is dominated by the chartists ($w^{(f)} = 0$). As remarked in the previous section, when the (chartist) risk aversion coefficient $\lambda^{(c)}$ is decreased (i.e. the parameter $\gamma = 1/(\lambda^{(c)} v(0))$ is increased) the fundamental steady state F changes from a stable (*Fig. 3a*) to an unstable focus (*Fig. 3b*) through a (supercritical) Neimark-Sacker bifurcation, which creates a stable closed orbit. The amplitude of the oscillations becomes wider for lower risk aversion (*Fig. 3c*) until the attractor "collapses" onto a stable non-fundamental steady state NF, with a permanent deviation of the price away from the fundamental (*Fig. 3d*). Similar phase-space transitions can be obtained by increasing the extrapolation parameter c or the speed of price adjustment β. The limit cycle and the non-fundamental steady state can be reached also starting from initial conditions with positive wealth proportion of the fundamentalists $w_0^{(f)} > 0$. An example is provided in *Fig. 4* (same parameter selection and initial values as *Fig. 3d*) which shows the phase-plot of a trajectory starting with $w_0^{(f)} = 0.56$ and attracted by the non-fundamental equilibrium represented in *Fig. 3d.* As a further example, it turns out that starting from $w_0^{(f)} = 0.45$ in the situation of *Fig. 3c*, the market ends up in the limit cycle shown there.

An important question concerns the role of the initial condition in situations similar to the one depicted in *Fig. 4*. Indeed, it can happen that the market price reaches the fundamental in the long-run, provided that a slightly different initial condition is chosen, for instance $w_0^{(f)} = 0.57$ in the case of *Fig. 4*.[19] This case is considered in *Fig. 5a*, which represents y_t versus time under two slightly different initial conditions. Thus the stable non-fundamental steady state of this example coexists in the phase space with a continuum of attracting fundamental steady states.[20]. The phase space is shared between the basins of attractions of different coexisting attractors. This is analyzed in greater detail in *Figs. 5b,c*. *Fig. 5b* represents the basins of attraction associated with the numerical example of *Fig. 5a*: the basins of the fundamental and non-fundamental steady states are obtained by allowing the initial values of the fundamentalist wealth share $w_0^{(f)}$ and the fundamental/price ratio y_0 to vary in the plane $w^{(f)}$- y, for fixed initial values of the other dynamic variables, and by representing in light grey (dark grey) the set of initial points which generate trajectories converging to the fundamental equilibria (non-fundamental equilibrium). Of course, the basins' structure depends on the particular parameter set used in the simulation. For instance, higher values of the chartist parameter c lead to an increase of the size of the basin of the non-fundamental steady state (compare *Fig. 5b*, where $c = 0.25$, with Fig. *5c*, where $c = 0.75$). *Figs. 5b,c* show that when the initial fundamentalist wealth proportion $w_0^{(f)}$ is sufficiently high, the system converges to the fundamental (no matter how far the initial price is from the fundamental).

[19] Similarly, under the same parameter selection and initial values of *Fig. 3c*, we find that for $w_0^{(f)} = 0.48$, the system converges to the fundamental steady state, while for $w_0^{(f)} = 0.45$ it reaches the limit cycle represented in *Fig. 3c*.

[20] The same is true, under a different choice of the parameters, for the limit cycle in *Fig. 3c*.

However, when $w_0^{(f)}$ is low, it is possible that the system is attracted by the non-fundamental steady state.[21]

***** FIGS. 3,4,5 approximately here *****

Fig. 6 shows that increasing values of the chartist extrapolation rate c can destabilize the price and produce a negative effect on fundamentalist wealth. The trajectories represented in *Figs. 6a, b, c* start from identical initial conditions, under increasing values of c. In all cases the price converges to the fundamental, but the increasing strength of extrapolation affects the nature and the length of the transient phase, and lowers the stationary level which is reached by the fundamentalist wealth proportion in the long-run (see the bifurcation diagram of $w^{(f)}$ versus c in *Fig. 6e*). Indeed, the higher is c, the longer is the transient characterized by price fluctuations around the fundamental, where fundamentalists' average profits are lower than chartists'. Notice also that when c becomes higher than a certain threshold[22], then the system is completely destabilized and no longer converges to the fundamental price but ends up in a limit cycle, with zero long-run fundamentalist wealth share and the market dominated by chartists (see the bifurcation diagrams of y and $w^{(f)}$ versus c in *Fig. 6d* and *6e*, respectively).

***** FIG. 6 approximately here *****

The out-of-equilibrium asymptotic behavior described so far represents situations where the fundamentalists are out of the market in the long-run, because their average profits are lower than chartists' profits. However, depending on the parameters, other attractors exist, where both types of agents survive in the long-run, with oscillatory behavior of the wealth shares. An example is the strange attractor $\mathcal{A}$ whose projection is represented in *Fig. 7a*. *Figs. 7b,c* show that the motion on the strange attractor has alternating phases, with the price much higher than fundamental when the market is dominated by chartists, whereas the fundamentalist wealth proportion increases when the price returns closer to the fundamental. Also in this case the attractor coexists with attracting fundamental steady states, as one can check by trying different initial conditions, and the role played by the initial wealth share $w_0^{(f)}$ seems to be particularly important, as in the case of *Fig. 5*. *Fig. 8* shows another example of an attractor which "allows" both groups to survive in the long-run, with time varying wealth shares. This is a periodic orbit $\mathcal{C}$ (*Fig. 8a*), where $w^{(f)}$ fluctuates approximately in the range $[38\%, 47\%]$ (*Fig. 8b*). Notice again the

[21] Quite surprisingly, this occurs when the initial price is not too far away from the fundamental, while in the opposite case the price returns to the fundamental. A possible explanation for this phenomenon is that in the latter case a higher fundamentalist demand (proportional to the relative deviation from the fundamental price) acts as a stronger mean reverting force.

[22] This threshold does not determine a bifurcation of the attractor. Indeed, changes of a parameter determine in general the shift of the boundary which separate the basins of different attractors. This is precisely what determines the bifurcation plots of *Figs 6d,e*, where the initial condition is assumed fixed.

role of the initial condition: for slightly different initial wealth shares, *Fig. 8b* shows two different trajectories, one attracted by the orbit, the other converging to a fundamental equilibrium, where the stationary fundamentalist wealth proportion is much higher.

***** FIGS. 7,8 approximately here *****

4.2 Stochastic dynamics

In this section we present simulation results concerning a simple stochastic version of the system (10)-(13). More precisely, it is assumed that the noisy dividend evolves according to $D_{t+1} = (1 + \phi + \sigma_\epsilon \epsilon_t) D_t$ (and thus the fundamental evolves according to $P^*_{t+1} = (1 + \phi + \sigma_\epsilon \ \epsilon_t) P^*_t$), where $\epsilon_t \sim \mathcal{N}(0, 1)$ are i.i.d. random shocks, and $\sigma_\epsilon > 0$ represents the standard deviation of the dividend (and fundamental) growth rate. As a consequence, the wealth of agent j evolves randomly according to

$$\Omega^{(j)}_{t+1} = \Omega^{(j)}_t \left[1 + r + Z^{(j)}_t \left(\frac{P_{t+1} + \chi_t (r - \phi) P^*_t - (1 + r) P_t}{P_t} \right) \right], \quad j \in \{f, c\},$$

where $\chi_t \equiv (1 + \phi + \sigma_\epsilon \epsilon_t)/(1 + \phi)$.

A stochastic term $P_t \sigma_\nu \nu_t$, where $\sigma_\nu > 0$ and $\nu_t \sim \mathcal{N}(0, 1)$ are i.i.d. random disturbances, is also added to the price setting equation (10)[23]. Introducing the usual changes of variables, we get the following noisy dynamical system:

$$\begin{aligned}
\rho_{t+1} &= \phi y_t + \beta Z_t + \sigma_\nu \nu_t \ , \\
y_{t+1} &= \frac{(1 + \phi + \sigma_\epsilon \epsilon_t)}{(1 + \rho_{t+1})} y_t \ , \\
m^{(c)}_{t+1} &= (1 - c) m^{(c)}_t + c \rho_{t+1}, \\
w^{(f)}_{t+1} &= \frac{w^{(f)}_t (1 + r + Z^{(f)}_t (\rho_{t+1} + \chi_t (r - \phi) y_t - r))}{(1 + r + Z_t (\rho_{t+1} + \chi_t (r - \phi) y_t - r))},
\end{aligned}$$

where:

$$\begin{aligned}
Z_t &= w^{(f)}_t Z^{(f)}_t + (1 - w^{(f)}_t) Z^{(c)}_t, \\
Z^{(f)}_t &= \frac{(\eta + r)(y_t - 1)}{\lambda^{(f)} \sigma^2_f}, \quad Z^{(c)}_t = \frac{\gamma}{\theta} \tanh[\theta (m^{(c)}_t + (r - \phi) y_t - r)],
\end{aligned}$$

and where ϵ_t and ν_t are i.i.d. processes, $\epsilon_t \sim \mathcal{N}(0, 1)$, $\nu_t \sim \mathcal{N}(0, 1)$, σ_ϵ, $\sigma_\nu > 0$, and $\chi_t = (1 + \phi + \sigma_\epsilon \epsilon_t)/(1 + \phi)$.

Fig. 9 represents sample paths of the prices (market price P_t and fundamental price P^*_t) and of the fundamentalist wealth share ($w^{(f)}_t$) as a function of time, under different selections of the parameters and different initial conditions.[24] In *Figs. 9a,b* and *9e,f*, the parameters are the same as in *Fig. 5a*

[23] In order to include, for instance, the effect of noise traders.

[24] The noise processes used in the simulation are obtained by setting $\sigma_\epsilon = 0.015$, $\sigma_\nu = 0.03$.

(coexistence of stable fundamental and non-fundamental steady states) while in *Figs. 9c,d*, the parameters are the same as in *Fig. 7* (coexistence of a strange attractor and stable fundamental steady state). In *Figs. 9a,b* and *9c,d*, the initial wealth share $w_0^{(f)}$ is sufficiently low. In both cases, the interaction of the nonlinear deterministic dynamics with external noise is able to generate phases of booms, where the price grows faster than the fundamental and the fundamentalist wealth share rapidly declines, followed by crashes, where the price is attracted again towards the fundamental, and the fundamentalist wealth share returns to higher levels. On the other hand, more regular price and wealth paths are obtained starting with a higher fundamentalist wealth proportion, as in the case represented in *Figs. 9e,f.* These phenomena may be related with the structure of the basins of attraction - and in particular with the role played by the initial wealth shares - of the underlying deterministic model.

We point out that the time series represented in Fig. 9 look rather realistic as compared with actual market behaviour, though the choice of the parameters and of the noise level used in these numerical experiments are not calibrated to real financial data. On the other hand the calibration of the model is naturally related to the question of the time scale: the model developed in the present paper is however agnostic with respect to the time scale, since the main focus of the paper has been on the qualitative price and wealth behaviour.

***** FIG. 9 approximately here *****

5 Conclusions and further research

Following the framework of Chiarella (1992), Chiarella et al (2002), Chiarella and He (2001a, 2003) and Brock and Hommes (1998), the interaction of fundamentalists and chartists has been incorporated in a market maker model of asset price and wealth dynamics. The resulting dynamical system for asset price and wealth turns out to be nonstationary, and a stationary system is developed by expressing the laws of motion in terms of capital gain, fundamental/price ratio and wealth proportions of the two types of agents. It is found that the presence of fundamentalists and chartists leads the stationary model to have two kinds of steady states, which often coexist in the phase-space, with different long-run stationary returns and wealth distributions: *fundamental* steady states, where the price is at the fundamental level, and *non-fundamental* steady states, where price grows faster than fundamental, while the fundamentalist wealth proportion becomes negligible in the long-run.

The chartists' extrapolation parameter c, together with the chartists' risk aversion $\lambda^{(c)}$ (inversely related to the slope γ of their demand function) and the market reaction coefficient β, play an important role in the local asymptotic stability of the fundamental steady states, and for sufficiently high values of c,

β, γ the price and return dynamics become unstable due to a Neimark-Sacker bifurcation.

The main impression gained from the numerical simulation of the global dynamics (Section 4) is that the model is able to generate a wide range of different dynamic scenarios, with a strong dependence on small changes of the parameters and of the initial conditions: limit cycles, periodic orbits, strange attractors, cases of coexistence of multiple steady states, or coexistence of a steady state with a cyclical attractor. In particular, in the case of coexistence of fundamental and non-fundamental steady states, the initial wealth distribution plays a crucial role in determining the long-run evolution.

Another important feature of this model is that it considers explicitly the interdependence between the price dynamics and the evolution of the wealth distribution among agent-types: it is found that in general fundamentalists' average profits are lower than chartists' profits (and thus the fundamentalist wealth proportion tends to vanish) when the system moves on a limit cycle or is at a nonfundamental steady state; on the other hand both types of agents survive in the long-run when the market is at a fundamental equilibrium, or when it fluctuates on periodic orbits or strange attractors. Anyway, in general the fundamentalists do not accumulate more wealth than the chartists.

We have also considered a stochastic version of the model, simulations of which have shown how the coexistence of fundamental and non fundamental equilibria, as well as the existence of cyclical attractors, can provide a basis for the onset of booms and crashes when random disturbances are added to the deterministic model. We have also observed that, because of this switching back and forth between the fundamental and non-fundamental equilibria, both groups seem to continue to survive with positive wealth shares. Thus at least in the simple framework of this paper we can see that supposed "irrational" traders (our chartists) can indeed survive in the long-run. Debate on this point in the economics and finance literature goes back at least to Friedman (1953) and Fama (1965).

Our analysis in this paper is based on a simplified model, and some extensions are needed in order to develop a more realistic one. First, the analysis here has focused mainly on a deterministic dynamic model which can be interpreted as the deterministic skeleton of a stochastic model with a noisy dividend process: our analysis of a stochastic version of the model could still be regarded as very preliminary, it would be interesting to analyze in greater detail the interaction of a noisy dividend process with the underlying deterministic scenarios. Second, although the dynamic modelling of the wealth proportions "keeps track" of realized profits of the two types of agents and determines endogenously time varying "weights" of fundamentalists and chartists, this model is one with fixed agents' proportions, in the sense that agents do not switch amongst different strategies on the basis on their realized profits or wealth (according to the adaptive belief system introduced by Brock and Hommes (1997a, 1998)). The introduction of "switching" mechanisms and time varying proportions (similar to Chiarella and He (2002)) would be an important extension of this model. Third, the introduction of a more flexible and realistic price setting rule, where the market maker inventory position is also taken into account, is likely to lead to more realistic dynamics of returns and wealth fractions.

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Appendix 1. The dynamic model under positive supply of shares

In this Appendix we show how the model can be extended to the case of positive net supply of shares. Similarly to Brock and Homes (1998), we first derive a benchmark notion of "fundamental solution", which refers to the long-run market clearing price path that would be obtained if agents were homogeneous with regard to their expectation of the excess return. Furthermore, this price is assumed to satisfy the "no-bubbles" condition.

Denote by $N_t^D = (\Omega_t^{(f)} Z_t^{(f)} + \Omega_t^{(c)} Z_t^{(c)})/P_t$ the demand (number of shares) and by N_t^S the supply of shares at t. The market clearing condition at time t, $N_t^D = N_t^S$, can be rewritten as

$$\sum_{j\in\{f,c\}} \Omega_t^{(j)} \frac{E_t^{(j)}[P_{t+1} + D_{t+1} - (1+r)P_t]}{\lambda^{(j)} Var_t^{(j)}[\rho_{t+1} + \delta_{t+1} - r]} = N_t^S P_t^2. \tag{17}$$

Let us assume that agents have constant (not necessarily homogeneous) beliefs about the variance of the (excess) return, i.e. $\sigma_{j,t}^2 \equiv Var_t^{(j)}[\rho_{t+1}+\delta_{t+1}-r] = \sigma_j^2$, $j \in \{f,c\}$. Denote by $\Omega_t = \sum_j \Omega_t^{(j)}$ the total wealth and by $w_t^{(j)} = \Omega_t^{(j)}/\Omega_t$ the wealth proportion of group j, with $w_t^{(c)} = 1 - w_t^{(f)}$. If all agents were homogeneous with regard to their expectation of the excess return, then eq. (17) could be rewritten as

$$E_t[P_{t+1} + D_{t+1}] = (1 + r + Q_t\xi_t)P_t, \tag{18}$$

where $Q_t \equiv N_t^S P_t/\Omega_t$ is the value of the supply of shares over total agents' wealth, while the quantity

$$\xi_t = \left(\sum_j w_t^{(j)} \frac{1}{\lambda^{(j)}\sigma_j^2}\right)^{-1} = \left[w_t^{(f)} \frac{1}{\lambda^{(f)}\sigma_f^2} + (1 - w_t^{(f)}) \frac{1}{\lambda^{(c)}\sigma_c^2}\right]^{-1}$$

is a time-varying weighted harmonic mean of the quantities $\lambda^{(j)}\sigma_j^2$, $j \in \{f, c\}$, that can be interpreted as an "average" risk perception. Thus the quantity $\pi_t \equiv Q_t \xi_t$ in eq. (18) would represent the *risk premium* required by the community of traders to hold the risky asset, under the assumed homogeneous beliefs, and $r_t^* \equiv r + Q_t \xi_t$ would be the required expected return, as perceived at time t. Eq. (18) is an expectational dynamic equation in the price, in which the required risk premium at time t, as one would expect, is a function of average risk attitudes and beliefs about risk. In the case of zero net supply ($N_t^S = 0$), eq. (18) reduces to the "no-arbitrage" equation (2), which has been discussed in Section 2.2. Here we sketch the structure of the model in the case of positive supply, assuming that agents are homogeneous with regard to risk aversion and beliefs about the variance. Moreover, we make the simplifying assumption that in a financial market with growing wealth and price processes, the value of the supply is a constant fraction Q of total wealth, $Q_t \equiv N_t^S P_t/\Omega_t = Q$, $\forall\, t$. A more general case, with heterogeneous risk beliefs and attitudes, sketched in Chiarella et al. (2004), is left to future research.

Let us derive the reference notion of fundamental solution. Denote by λ and by σ^2 the common risk aversion and belief about the variance of the excess return, respectively. Then the (constant) required risk premium $\pi_t = \pi$ and the required expected return $r_t^* = r^*$ turn out to be $\pi \equiv Q\lambda\sigma^2$, $r^* = r + \pi = r + Q\lambda\sigma^2$, and the market clearing condition (17) yields $E_t[P_{t+1} + D_{t+1}] = (1 + r^*)P_t$. Assuming homogeneous beliefs about expected dividends and a dividend process which evolves according to $E_t[D_{t+k}] = (1+\phi)^k D_t$, $k = 1, 2, ...$, $\phi \geq 0$, the unique fundamental solution P_t^* is given by

$$P_t^* = \frac{(1+\phi)D_t}{(r^* - \phi)} = \frac{(1+\phi)D_t}{(r + Q\lambda\sigma^2 - \phi)}, \tag{19}$$

where P_t^* evolves over time according to $E_t[P_{t+1}^*] = (1 + \phi)P_t^*$. The expected dividend yield and capital gain along the fundamental solution are given by $E_t[\delta_{t+1}] = r^* - \phi$, $E_t[\rho_{t+1}] = \phi$, while the expected return is $r^* = r + Q\lambda\sigma^2 = E_t[\rho_{t+1}] + E_t[\delta_{t+1}]$.

Next, by introducing heterogeneous and time varying beliefs, and following similar steps as in the case of zero supply, we obtain the dynamical system

$$
\begin{aligned}
P_{t+1} &= P_t + \phi P_t^* + P_t \beta (Z_t - Q), \\
m_{t+1}^{(c)} &= (1-c) m_t^{(c)} + c[(P_{t+1} - P_t)/P_t], \\
P_{t+1}^* &= (1+\phi) P_t^*, \\
\Omega_{t+1}^{(j)} &= \Omega_t^{(j)} \left[1 + r + Z_t^{(j)} \left(\frac{P_{t+1} + D_{t+1} - (1+r)P_t}{P_t} \right) \right], \quad j \in \{f, c\},
\end{aligned}
$$

where:

$$
\begin{aligned}
\Omega_t &= \Omega_t^{(f)} + \Omega_t^{(c)}, \quad Z_t = (\Omega_t^{(f)} Z_t^{(f)} + \Omega_t^{(c)} Z_t^{(c)})/\Omega_t, \\
Z_t^{(f)} &= \frac{\eta(P_t^* - P_t) + \phi P_t^* + (1+\phi) D_t - r P_t}{P_t \lambda \sigma^2}, \quad Z_t^{(c)} = \frac{m_t^{(c)} + (1+\phi) D_t / P_t - r}{\lambda \sigma_{c,t}^2},
\end{aligned}
$$

and the chartist variance belief $\sigma_{c,t}^2$ is assumed in general to be state-dependent, with stationary level $\sigma_{c,t}^2 = \sigma^2 (= \sigma_f^2)$ at the "fundamental solution". Notice that from (19) we get $(1+\phi)D_t = (r^* - \phi)P_t^*$, so that agents' demand functions can be rewritten as (recall also that $(r^* - r) = Q\lambda\sigma^2$)

$$
\begin{aligned}
Z_t^{(f)} &= \frac{(\eta + r)(P_t^* - P_t)/P_t + (r^* - r)P_t^*/P_t}{\lambda\sigma^2} = \frac{(\eta + r)(P_t^* - P_t)/P_t}{\lambda\sigma^2} + Q\frac{P_t^*}{P_t}, \\
Z_t^{(c)} &= \frac{m_t^{(c)} + (r^* - \phi)P_t^*/P_t - r}{\lambda\sigma_{c,t}^2} = \frac{m_t^{(c)} + (r - \phi)P_t^*/P_t - r}{\lambda\sigma_{c,t}^2} + \frac{Q\sigma^2}{\sigma_{c,t}^2}\frac{P_t^*}{P_t},
\end{aligned}
$$

A stationary system can be obtained through the same changes of variables used in the simplified zero-supply case, and similar results about the steady states hold. Notice that at the fundamental steady states (where $y_t = P_t^*/P_t = 1$, $m_t^{(c)} = \phi$, and where $\sigma_{c,t}^2 = \sigma_f^2 = \sigma^2$ under our assumptions) the total agents' demand is exactly equal to the supply, $\overline{Z}^{(c)} = \overline{Z}^{(f)} = \overline{Z} = Q = (r^* - r)/\lambda\sigma^2$.

Appendix 2. Analysis of the steady states

In this Appendix we derive the steady states of the model, and discuss the stability properties of the fundamental steady state.

Derivation of the steady states

First of all, notice that the subsets of the phase space of equation $w^{(f)} = 0$ (denote it by X^c) and $w^{(f)} = 1$ (denote it by X^f) are *trapping*, in the sense that $T(X^c) \subseteq X^c$ and $T(X^f) \subseteq X^f$, where T is the map (15); these trapping subsets represent the cases where only chartists or only fundamentalists survive in the market, respectively.

The steady states of the system are the fixed points $(\overline{\rho}, \overline{y}, \overline{m}^{(c)}, \overline{w}^{(f)})$ of the map T, obtained by setting $(\rho', y', m^{(c)\prime}, w^{(f)\prime}) = (\rho, y, m^{(c)}, w^{(f)}) = (\overline{\rho}, \overline{y}, \overline{m}^{(c)}, \overline{w}^{(f)})$

in (15). Thus the steady states must satisfy the following set of conditions

$$\begin{aligned}
\overline{\rho} &= \phi\overline{y} + \beta\overline{Z}, & (20)\\
\overline{y} &= \overline{y}(1+\phi)/(1+\overline{\rho}), & (21)\\
\overline{m}^{(c)} &= (1-c)\overline{m}^{(c)} + c\overline{\rho}, & (22)\\
\overline{w}^{(f)} &= \overline{w}^{(f)}\frac{1+r+\overline{Z}^{(f)}(\overline{\rho}+(r-\phi)\overline{y}-r)}{1+r+\overline{Z}(\overline{\rho}+(r-\phi)\overline{y}-r)}, & (23)
\end{aligned}$$

where $\overline{Z} = \overline{w}^{(f)}\overline{Z}^{(f)} + (1-\overline{w}^{(f)})\overline{Z}^{(c)}$, and

$$\overline{Z}^{(f)} = \frac{(\eta+r)(\overline{y}-1)}{\lambda^{(f)}\sigma_f^2}, \quad \overline{Z}^{(c)} = \frac{\gamma}{\theta}\tanh[\theta(\overline{m}^{(c)} + (r-\phi)\overline{y} - r)].$$

Notice first that (22) implies $\overline{m}^{(c)} = \overline{\rho}$. In the following we consider three separate cases, $\overline{w}^{(f)} = 0$, $\overline{w}^{(f)} = 1$, and $0 < \overline{w}^{(f)} < 1$.

(i) We first consider the case $\overline{w}^{(f)} = 0$, i.e. we look for the fixed points of the restriction of the map T to the subset X^c. Therefore we neglect eq. (23) and we set $\overline{Z} = \overline{Z}^{(c)}$ in (20). Assume first $\overline{y} > 0$. Then (21) implies $\overline{\rho} = \phi$ $(= \overline{m}^{(c)})$, and (20) becomes

$$\phi(\overline{y}-1) + \beta\frac{\gamma}{\theta}\tanh[\theta(r-\phi)(\overline{y}-1)] = 0. \quad (24)$$

Since $(r-\phi)$ is positive, it follows that the terms $\phi(\overline{y}-1)$ and $(\beta\gamma/\theta)\tanh[\theta(r-\phi)(\overline{y}-1)]$ are both positive for $\overline{y} > 1$, both negative for $\overline{y} < 1$, and therefore condition (24) yields $\overline{y} = 1$. We denote by "fundamental" steady state the one characterized by $\overline{y} = 1$, $\overline{\rho} = \overline{m}^{(c)} = \phi$.

Now assume $\overline{y} = 0$. In this case (20) becomes

$$\frac{\overline{\rho}}{\beta} = \frac{\gamma}{\theta}\tanh[\theta(\overline{\rho}-r)]. \quad (25)$$

Since $g_1(\overline{\rho}) \equiv \overline{\rho}/\beta$ is a straight line through the origin with positive slope $1/\beta$, and $g_2(\overline{\rho}) \equiv (\gamma/\theta)\tanh[\theta(\overline{\rho}-r)]$ is a strictly increasing S-shaped function, taking values in $(-\gamma/\theta, \gamma/\theta)$ and vanishing for $\overline{\rho} = r$, it follows that a negative solution of (25) always exists, while (25) admits one or two further positive solutions provided that γ or β are sufficiently high (i.e. for low chartist risk aversion or strong price reaction). Moreover, if a positive solution $\overline{\rho}$ exists, it necessarily follows that $\overline{\rho} > r$, and therefore $\overline{\rho} > \phi$. We denote by "non-fundamental" steady states the ones characterized by $\overline{y} = 0$, $\overline{\rho} = \overline{m}^{(c)} > \phi$, where $\overline{\rho}$ is a positive solution of (25).

(ii) Let us now consider the case $\overline{w}^{(f)} = 1$, i.e. look for the fixed points of the restriction of the map T to the subset X^f. Following similar steps as in the previous case, one easily finds that two steady states exist, a fundamental

steady state with $\overline{y}=1$, $\overline{\rho}=\phi$, and a further steady state $\overline{y}=0$, $\overline{\rho}=-\beta(\eta+r)/(\lambda^{(f)}\sigma_f^2)<0$.

(iii) We now consider the case $0<\overline{w}^{(f)}<1$, i.e. we look for fixed points characterized by strictly positive stationary wealth shares of fundamentalists and chartists. Assume first $\overline{y}>0$. Then (21) implies $\overline{\rho}=\overline{m}^{(c)}=\phi$, and (20) becomes

$$\left(\phi+\beta\overline{w}^{(f)}\frac{\eta+r}{\lambda^{(f)}\sigma_f^2}\right)(\overline{y}-1)+\beta(1-\overline{w}^{(f)})\frac{\gamma}{\theta}\tanh[\theta(r-\phi)(\overline{y}-1)]=0, \quad (26)$$

which yields $\overline{y}=1$. Therefore $\overline{Z}^{(f)}=\overline{Z}^{(c)}=\overline{Z}=0$ and (23) is identically satisfied for any $\overline{w}^{(f)}$, $0<\overline{w}^{(f)}<1$. Thus, a continuum of fundamental steady states exist.

Assume now $\overline{y}=0$. One can show that in general no fixed points of (15) exist with $\overline{y}=0$, $0<\overline{w}^{(f)}<1$. In fact from (23) such a fixed point would imply

$$\overline{Z}^{(f)}(\overline{\rho}-r)=\overline{Z}^{(c)}(\overline{\rho}-r), \quad (27)$$

where $\overline{Z}^{(f)}=-(\eta+r)/(\lambda^{(f)}\sigma_f^2)$, $\overline{Z}^{(c)}=(\gamma/\theta)\tanh[\theta(\overline{\rho}-r)]$. On the other hand (20) would become

$$\frac{\overline{\rho}}{\beta}=\overline{w}^{(f)}\overline{Z}^{(f)}+(1-\overline{w}^{(f)})\overline{Z}^{(c)}. \quad (28)$$

Eq. (28) implies $\overline{\rho}\neq r$ (otherwise the left-hand side and the right-hand side of (28) would have opposite sign) and therefore (27) implies $\overline{Z}^{(f)}=\overline{Z}^{(c)}$, i.e.

$$-\frac{\eta+r}{\lambda^{(f)}\sigma_f^2}=\frac{\gamma}{\theta}\tanh[\theta(\overline{\rho}-r)], \quad (29)$$

while (28) becomes

$$\frac{\overline{\rho}}{\beta}=\overline{Z}^{(c)}=\frac{\gamma}{\theta}\tanh[\theta(\overline{\rho}-r)],$$

which has been discussed in the previous case *(i)* and which in general will not be compatible with (29).

To summarize, if we restrict our analysis to the steady states which are characterized by $\overline{\rho}\geq 0$[25], we can identify two types of steady states:

a) "fundamental" steady states, characterized by $\overline{y}=1$, $\overline{\rho}=\overline{m}^{(c)}=\phi$, any $\overline{w}^{(f)}\in[0,1]$, and $\overline{Z}^{(f)}=\overline{Z}^{(c)}=\overline{Z}=0$, i.e. by the price being at the fundamental and growing at the fundamental rate, any long-run wealth distribution, and zero excess demand. As already remarked, these represent a *continuum* of steady states.

[25] Though the existence of steady states with $\overline{\rho}<0$ has been proven analytically, they are in general outside the economically meaningful range of values of ρ (i.e. $\rho>-1$) for reasonable values of the parameters, and numerical evidence confirms that they are not attracting.

b) “non fundamental” steady states, characterized by $\overline{y} = 0$, $\overline{w}^{(f)} = 0$, $\overline{\rho} = \overline{m}^{(c)} > r > \phi$, where $\overline{\rho}$ solves $\overline{\rho}/\beta = (\gamma/\theta)\tanh[\theta(\overline{\rho} - r)]$, and $\overline{Z} = \overline{Z}^{(c)} = (\gamma/\theta)\tanh[\theta(\overline{\rho} - r)] > 0$, i.e. by the price growing faster than the fundamental, the fundamental/price ratio approaching zero, market dominated by chartists, and permanent positive excess demand.

Local stability conditions of the fundamental steady states

In order to gain some insights about the conditions of local asymptotic stability of the “fundamental steady states”, and their dependence on the key parameters of the model, we restrict our analysis to the particular cases where only fundamentalists or only chartists populate the market, i.e. the cases in which the dynamical system is restricted to the subspace X^f ($w^{(f)} = 1$) or X^c ($w^{(f)} = 0$), respectively. The analysis of the local stability at the continuum of steady states with $0 < \overline{w}^{(f)} < 1$ would become much more difficult to work out, given the higher dimension of the dynamical system in this case and the dependence of the Jacobian on the stationary wealth level $\overline{w}^{(f)}$. On the other hand, the analysis of the extreme cases is quite illuminating about the role played by the key parameters in stabilizing or destabilizing the steady state, and numerical simulations confirm that the sensitivity to the parameters in the general case is similar to what emerges from the particular cases $w^{(f)} = 1$ and $w^{(f)} = 0$.

(a) The map $T^{(f)} : (\rho, y) \longmapsto (\rho', y')$ which drives the dynamical system restricted to X^f is

$$T^{(f)} : \begin{cases} \rho' = \phi y + \beta Z^{(f)}, \\ y' = y(1+\phi)/(1+\rho'), \end{cases}$$

where $Z^{(f)} = (\eta + r)(y-1)/(\lambda^{(f)}\sigma_f^2)$ and the dynamic equation for $m^{(c)}$ has been neglected given that it has no influence on the dynamics of ρ and y. The map $T^{(f)}$ can indeed be reduced to the one-dimensional map with equation $y' = F(y)$, where

$$F(y) = y(1+\phi)\left[1 + \phi y + \beta\frac{(\eta+r)(y-1)}{\lambda^{(f)}\sigma_f^2}\right]^{-1}.$$

Given that $y = 1$ at the fundamental steady state, one easily finds

$$\left.\frac{dF(y)}{dy}\right|_{y=1} = \frac{1 - \beta(\eta+r)/(\lambda^{(f)}\sigma_f^2)}{(1+\phi)} < 1$$

Thus the local asymptotic stability condition $-1 < \left.\frac{dF(y)}{dy}\right|_{y=1} < 1$ reduces to $\beta(\eta+r)/(\lambda^{(f)}\sigma_f^2) < 2 + \phi$. Moreover, for $\beta(\eta+r)/(\lambda^{(f)}\sigma_f^2) = 2 + \phi$ the stable steady becomes unstable through a Flip-bifurcation.

(b) The map $T^{(c)} : (\rho, y, m^{(c)}) \longmapsto (\rho', y', m^{(c)\prime})$ which governs the dynamical system restricted to X^c is

$$T^{(c)} : \begin{cases} \rho' = \phi y + \beta Z^{(c)}, \\ y' = y(1+\phi)/(1+\rho'), \\ m^{(c)\prime} = (1-c)m^{(c)} + c\rho', \end{cases}$$

where $Z^{(c)} = (\gamma/\theta)\tanh[\theta(m^{(c)} + (r-\phi)y - r)]$. This map could in fact be immediately reduced to a two-dimensional map, since $\rho^{'}$ is a function of y, $m^{(c)}$, but one can easily handle the $3-D$ specification as well. The Jacobian matrix of $T^{(c)}$, evaluated at the fundamental steady state F (where $\rho = m^{(c)} = \phi$, $y = 1$) is given by

$$DT^{(c)}(F) = \begin{bmatrix} 0 & \phi + \beta\gamma(r-\phi) & \beta\gamma \\ 0 & [1 - \beta\gamma(r-\phi)]/(1+\phi) & -\beta\gamma/(1+\phi) \\ 0 & c[\phi + \beta\gamma(r-\phi)] & (1-c) + c\beta\gamma \end{bmatrix},$$

which implies that one eigenvalue is 0 (and thus smaller than one in modulus), while the remaining two eigenvalues are the ones of the lower-right two-dimensional block of $DT^{(c)}(F)$ (denote it by A). Let

$$\begin{aligned} Tr(A) &= 1 - \frac{\phi + \beta\gamma(r-\phi)}{1+\phi} + (1-c) + c\beta\gamma, \\ Det(A) &= (1-c)\left[1 - \frac{\phi + \beta\gamma(r-\phi)}{1+\phi}\right] + c\beta\gamma, \end{aligned}$$

be the trace and the determinant of A, respectively. The characteristic polynomial of A is given by $\mathcal{P}(z) = z^2 - Tr(A)z + Det(A)$. A well known necessary and sufficient condition (see e.g. Gumowski and Mira (1980), Medio and Lines (2001)) for both eigenvalues of A to be smaller than one in modulus, which implies a locally attracting steady state, is

$$\begin{cases} \mathcal{P}(1) = 1 - Tr(A) + Det(A) > 0, \\ \mathcal{P}(-1) = 1 + Tr(A) + Det(A) > 0, \\ \mathcal{P}(0) = Det(A) < 1. \end{cases} \tag{30}$$

In terms of the parameters of the model (r, ϕ, c, and the aggregate parameter $\beta\gamma$) the set of inequalities (30) can be rewritten as

$$c\,[\phi + \beta\gamma(r-\phi)] > 0, \tag{31}$$

$$\beta\gamma[(2-c)(r-\phi) - 2c(1+\phi)] < (2-c)(2+\phi), \tag{32}$$

$$\beta\gamma[c(1+r) - (r-\phi)] < (c+\phi). \tag{33}$$

Condition (31) is always true under our assumptions about the parameters ($r > \phi$).

Since $0 < c < 1$, the right-hand side of (32) is positive, and this implies that (32) is satisfied when $[(2-c)(r-\phi) - 2c(1+\phi)] \leq 0$, i.e. $c \geq 2(r-\phi)/[(r-\phi)+$

$2(1+\phi)]$, while in the opposite case, $c < 2(r-\phi)/[(r-\phi)+2(1+\phi)]$, condition (32) is satisfied only for

$$\beta\gamma < \frac{(2-c)(2+\phi)}{2(r-\phi)-c[(r-\phi)+2(1+\phi)]}.$$

Finally, the right-hand side of (33) is positive, and this implies that (33) is satisfied when $c \leq (r-\phi)/(1+r)$, while in the opposite case $c > (r-\phi)/(1+r)$ condition (33) is satisfied only for

$$\beta\gamma < \frac{(c+\phi)}{c(1+r)-(r-\phi)}.$$

Taking r, ϕ as given, the region of local asymptotic stability and the bifurcation curves can be represented in the space of the parameters (c, Δ), where $\Delta \equiv \beta\gamma$[26], as qualitatively shown in *Fig. 2* (shaded area). In particular, when Δ or c are varied so that the bifurcation curve of equation $\Delta = \Delta_{NS}(c) = (c+\phi)/[c(1+r)-(r-\phi)]$ is crossed from inside the stability region (as shown in *Fig. 2*), then a *supercritical*[27] Neimark-Sacker bifurcation occurs, which is followed by the appearance of a *stable* limit cycle.

The stability region of *Fig. 2* shows that for Δ low enough ($\Delta \leq 1/(1+r)$) the fundamental steady state F is stable for any value of the chartist extrapolation parameter c, $0 < c < 1$, while in general a Neimark-Sacker bifurcation will occur when $\Delta > 1/(1+r)$ and c is increased above a certain threshold[28].

FIGURE CAPTIONS

Fig. 1

The assumed sigmoid shape of the chartist investment fraction in the risky asset $Z^{(c)}(x) = (\gamma/\theta)\tanh\{\theta x\}$, as a function of the expected excess return, for different values of the parameter $\gamma = 1/(\lambda^{(c)}v(0))$ (a) and θ (b). An example of the "implied" variance $v(x)$ is represented in (c).

Fig. 2

Qualitative sketch of the region of local asymptotic stability of the fundamental steady state for the map $T^{(c)}$, associated with the case $w^{(f)} = 0$ (no fundamentalists). The stability region (shaded area) is represented in the space of the parameters (c, Δ) where $\Delta \equiv \beta\gamma$. The equation of the Neimark-Sacker bifurcation curve is $\Delta = \Delta_{NS}(c) = (c+\phi)/[c(1+r)-(r-\phi)]$.

[26] The aggregate parameter $\Delta \equiv \beta\gamma$ is the partial derivative of ρ' with respect to $m^{(c)}$ (evaluated at the steady state F) and can be interpreted as the sensitivity of tomorrow's return with respect to today's chartist expected return.

[27] We have numerical evidence of the supercritical nature of the Neimark-Sacker bifurcation.

[28] The branch of the Flip boundary qualitatively represented in *Fig. 2* is of no practical interest in this case, because it is related to very low values of c ($0 < c < 2(r-\phi)/[(r-\phi)+2(1+\phi)] \simeq 0.009852$ in our numerical examples).

Fig. 3

The case of a market without fundamentalists ($w^{(f)} = 0$). Effect of decreasing the chartist risk aversion $\lambda^{(c)}$ (i.e. increasing $\gamma \equiv 1/(\lambda^{(c)}v(0))$) under the same initial condition $\rho_0 = 0.01$, $m_0^{(c)} = 0.015$, $y_0 = 0.9$. The parameters are: $v(0) = 0.002$, $\theta = 100$, $c = 0.25$, $\beta = 0.05$, $r = 0.02$, $\phi = 0.01$.

Fig. 4

Convergence to a non-fundamental equilibrium where the price grows faster than the fundamental, with $\overline{\rho} \simeq 3.428\% > \phi = 1\%$ (a), and only the chartists survive (b). The initial condition is the same as *Fig. 3*, but in this case $w_0^{(f)} = 56\%$. The parameters are the same as *Fig. 3d*, with $\eta = 0.3$, $\lambda^{(f)} = 20$, $\sigma_f^2 = 0.002$.

Fig. 5

Role of initial condition (wealth shares and fundamental/price ratio) in the long-run dynamics. (a) the trajectory generated by the initial condition $\rho_0 = 0.01$, $m_0^{(c)} = 0.015$, $y_0 = 0.9$, $w_0^{(f)} = 56\%$ converges to a non-fundamental steady state where only chartists survive. If the initial wealth share $w_0^{(f)}$ is slightly increased to 57%, the system converges to a fundamental steady state with positive stationary wealth shares of both groups. The parameters are as in *Fig. 4*, i.e. $\eta = 0.3$, $\lambda^{(f)} = 20$, $\sigma_f^2 = 0.002$, $\lambda^{(c)} = 6.5$, $v(0) = 0.002$ (and therefore $\gamma = 1/(\lambda^{(c)}v(0)) \simeq 76.923$), $\theta = 100$, $c = 0.25$, $\beta = 0.05$, $r = 0.02$, $\phi = 0.01$. (b),(c) role of initial wealth shares and fundamental/price ratio in the long-run dynamics, captured through the graphical representation of the basins of attraction of the fundamental and non-fundamental steady states. Note the dependence on the chartist strength of extrapolation c, by comparing (b), where $c = 0.25$ with (c), where $c = 0.75$. The other parameters are the same as in (a).

Fig. 6

(a), (b), (c) show the effect of the chartist extrapolation rate (c) on transient and long-run dynamics of the fundamental/price ratio, under the same initial conditions $\rho_0 = 0.01$, $m_0^{(c)} = 0.015$, $y_0 = 0.8$, $w_0^{(f)} = 40\%$. The other parameters are: $\eta = 0.1$, $\lambda^{(f)} = 10$, $\sigma_f^2 = 0.004$, $\lambda^{(c)} = 7$, $v(0) = 0.004$ (and therefore $\gamma \simeq 35.714$), $\theta = 50$, $\beta = 0.05$, $r = 0.02$, $\phi = 0.01$. Under the same fixed initial conditions, the bifurcation plots (d) and (e) represent the long-run dynamics of the fundamental/price ratio and the fundamentalist wealth share, respectively, as a function of c.

Fig. 7

A strange attractor, whose cyclical movements of prices and wealth shares coexist with the stable fundamental equilibrium. (a) projection of the attractors in the (ρ, y)-plane . (b), (c) phases where the price is much higher than the fundamental (low fundamental/price ratio) are dominated by chartists (in terms of wealth shares). The parameters are: $\eta = 0.3$, $\lambda^{(f)} = 20$, $\sigma_f^2 = 0.002$, $\lambda^{(c)} = 7$,

$v(0) = 0.002$ ($\gamma = 1/(\lambda^{(c)}v^{(c)}(0)) \simeq 71.429$), $\theta = 100$, $c = 0.25$, $\beta = 0.05$, $r = 0.02$, $\phi = 0.01$.

Fig. 8

An attracting periodic orbit coexists with the stable fundamental equilibrium. (a) projection of the two attractors in the (ρ, y)-plane. (b) the trajectory generated by the initial conditions $\rho_0 = m_0^{(c)} = 0.01$, $y_0 = 0.85$, $w_0^{(f)} = 40\%$ converges to the fundamental steady state after a very long transient. If the initial wealth share $w_0^{(f)}$ is decreased to 38%, the system converges to the periodic orbit, with long-run fluctuations in wealth shares. Here the parameter set is characterized by strong fundamentalist reaction ($\eta = 0.8$) and strong chartist extrapolation ($c = 0.8$). The other parameters are: $\lambda^{(f)} = 10$, $\sigma_f^2 = 0.004$, $\lambda^{(c)} = 2.5$, $v(0) = 0.004$ ($\gamma = 100$), $\theta = 50$, $\beta = 0.1$, $r = 0.02$, $\phi = 0.01$.

Fig. 9

Time series of the price P_t and the fundamental P_t^* (a,c,e), and related time series of the fundamentalist wealth share $w_t^{(f)}$ (b,d,f) in stochastic simulations associated with different underlying deterministic dynamics. Trajectories start with $\rho_0 = 0.01$, $m_0^{(c)} = 0.015$, $y_0 = 0.9$. In (a)-(b) and (e)-(f) the parameters are the same as in *Fig. 5a,b*, but the initial fundamentalist wealth share is much higher in (f) ($w_0^{(f)} = 60\%$) than in (b) ($w_0^{(f)} = 30\%$). In (c),(d) the parameters are the same as in *Fig. 7*, and $w_0^{(f)} = 30\%$. Booms and crashes of the price similar to (a) and (c) are often observed when $w_0^{(f)}$ is sufficiently low and external noise is added to the system, which may be related to the coexistence of attractors and the structure of the basins of the underlying deterministic model. The assumed noise processes are characterized by $\sigma_\epsilon = 0.015$ (dividend growth rate) and $\sigma_\nu = 0.03$ (price adjustment).

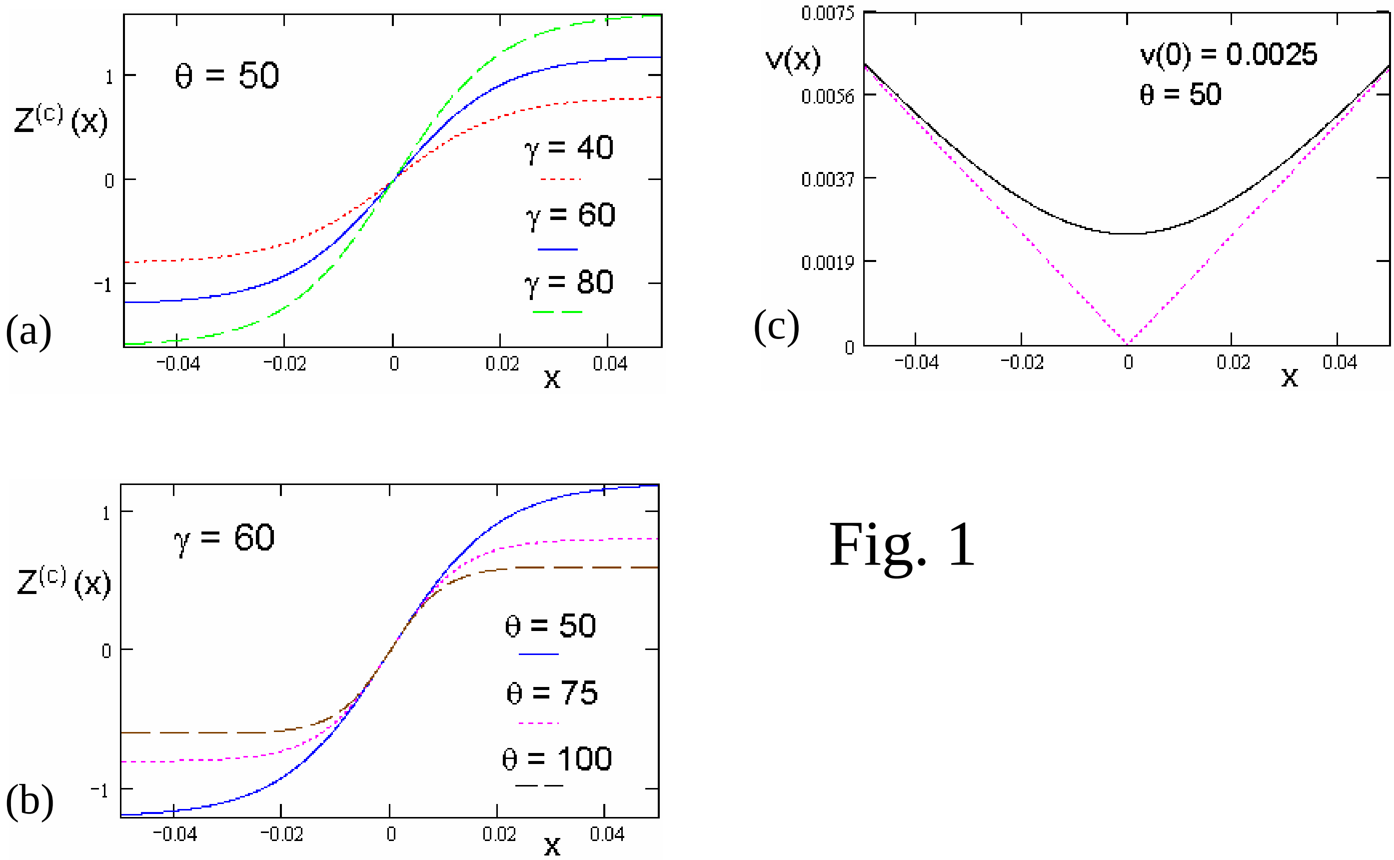
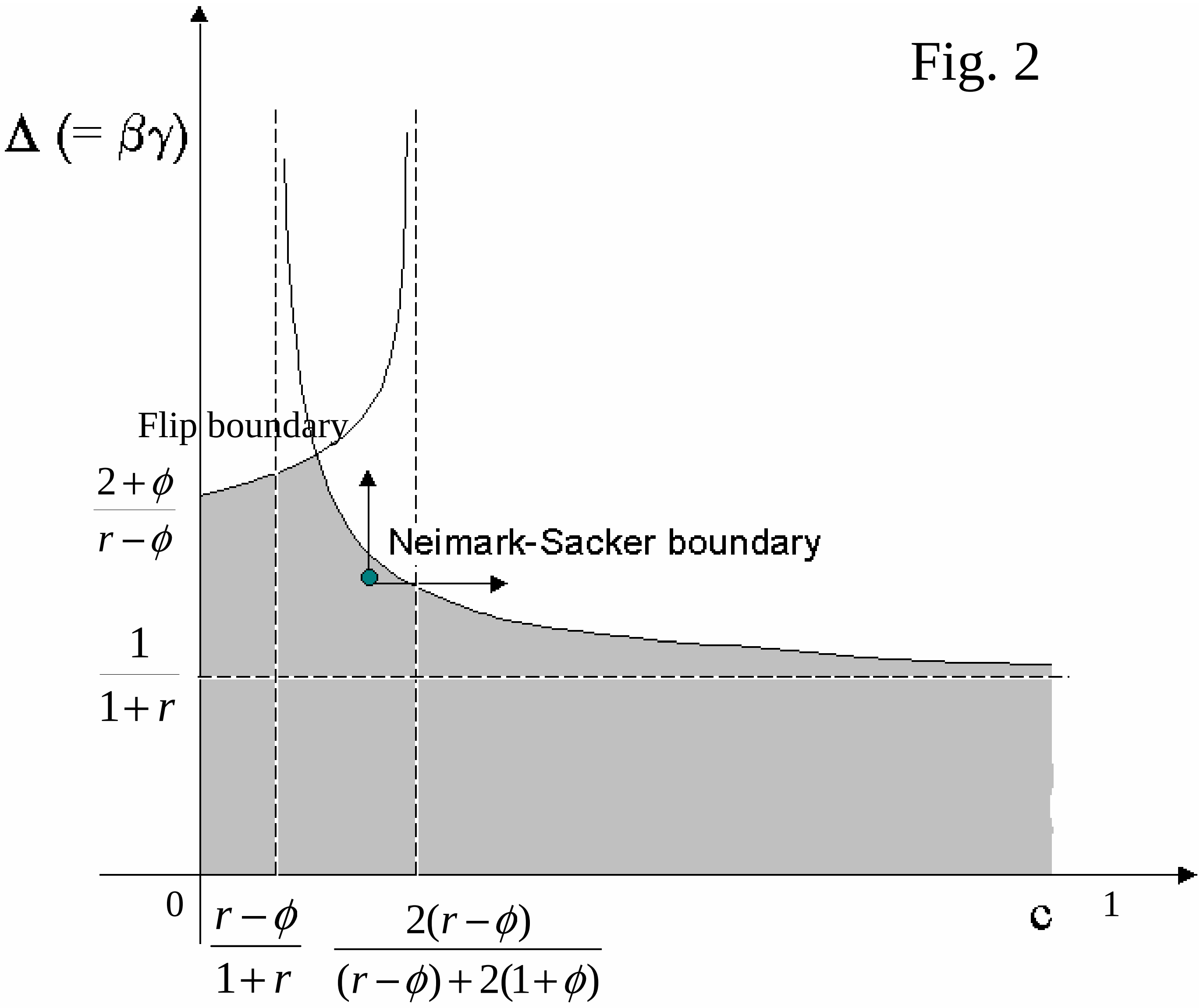


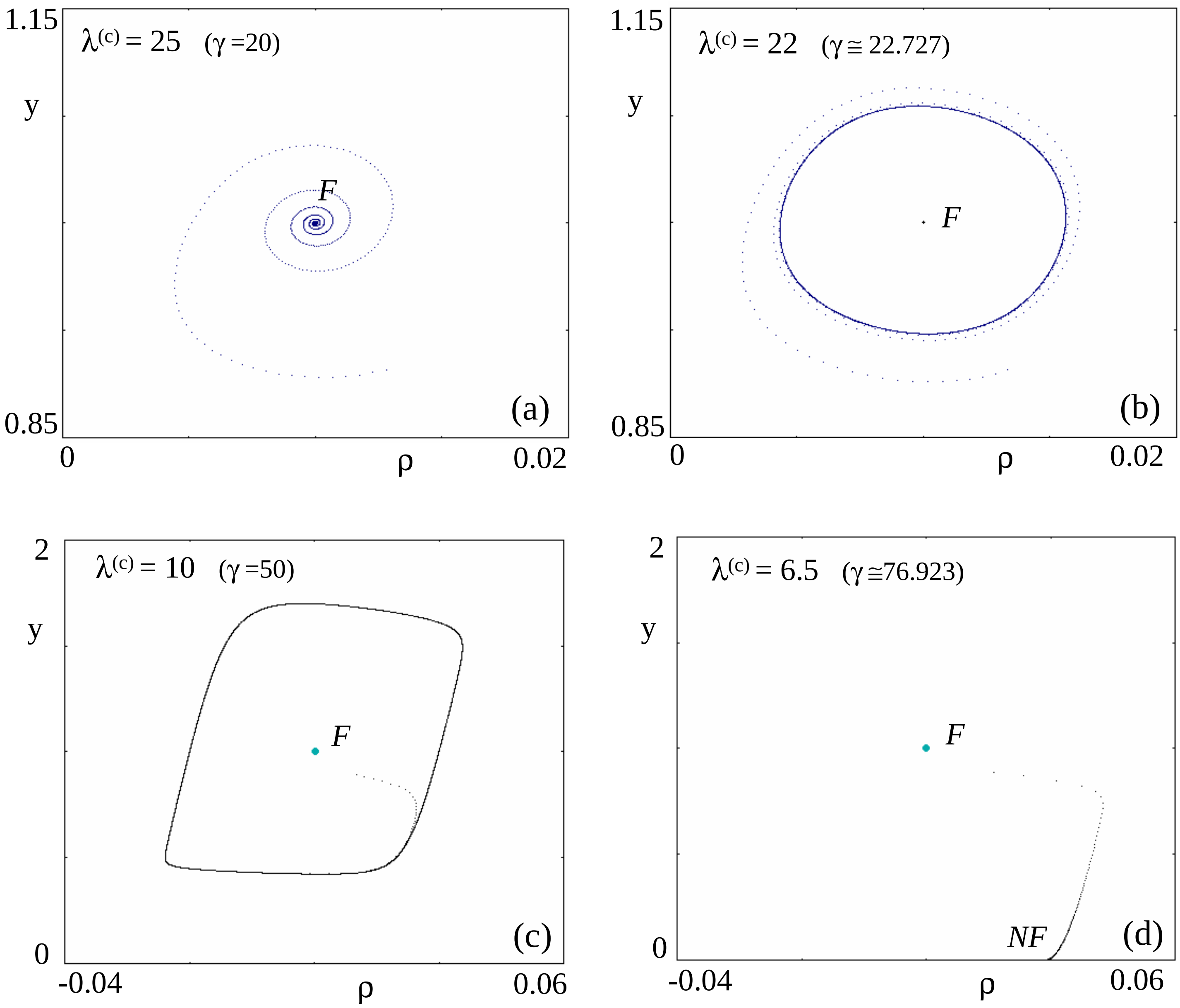
Fig. 1

Fig. 2



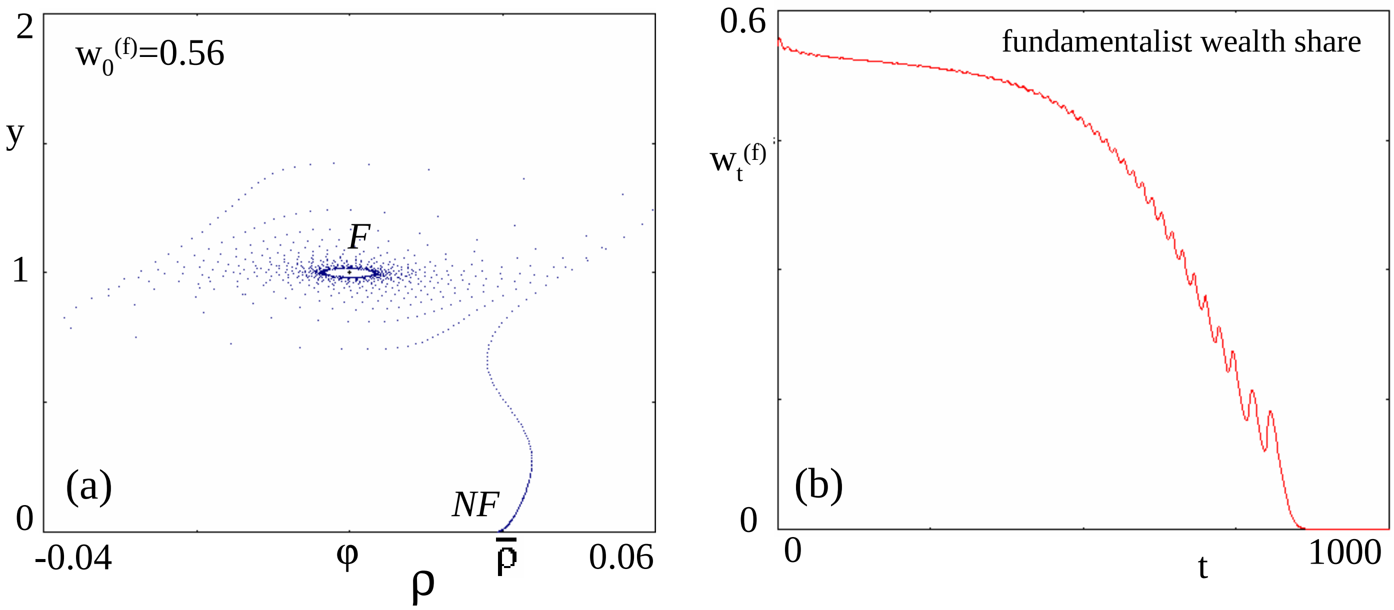
$\Delta \ (= \beta\gamma)$
Flip boundary
$\frac{2+\phi}{r-\phi}$
Neimark-Sacker boundary
$\frac{1}{1+r}$
0
$\frac{r-\phi}{1+r}$
$\frac{2(r-\phi)}{(r-\phi)+2(1+\phi)}$
c
1

Fig. 3



1.15
$\lambda^{(c)} = 25$ (γ =20)
y
F
(a)
0.85
0
ρ
0.02
1.15
$\lambda^{(c)} = 22$ ($\gamma \cong 22.727$)
y
F
(b)
0.85
0
ρ
0.02
2
$\lambda^{(c)} = 10$ (γ =50)
y
F
(c)
0
-0.04
ρ
0.06
2
$\lambda^{(c)} = 6.5$ ($\gamma \cong 76.923$)
y
F
NF
(d)
0
-0.04
ρ
0.06

Fig. 4



2
w0(f)=0.56
y
F
1
(a)
NF
0
-0.04
φ
ρ̄
0.06
ρ
0.6
fundamentalist wealth share
wt(f)
(b)
0
0
t
1000

Fig. 5

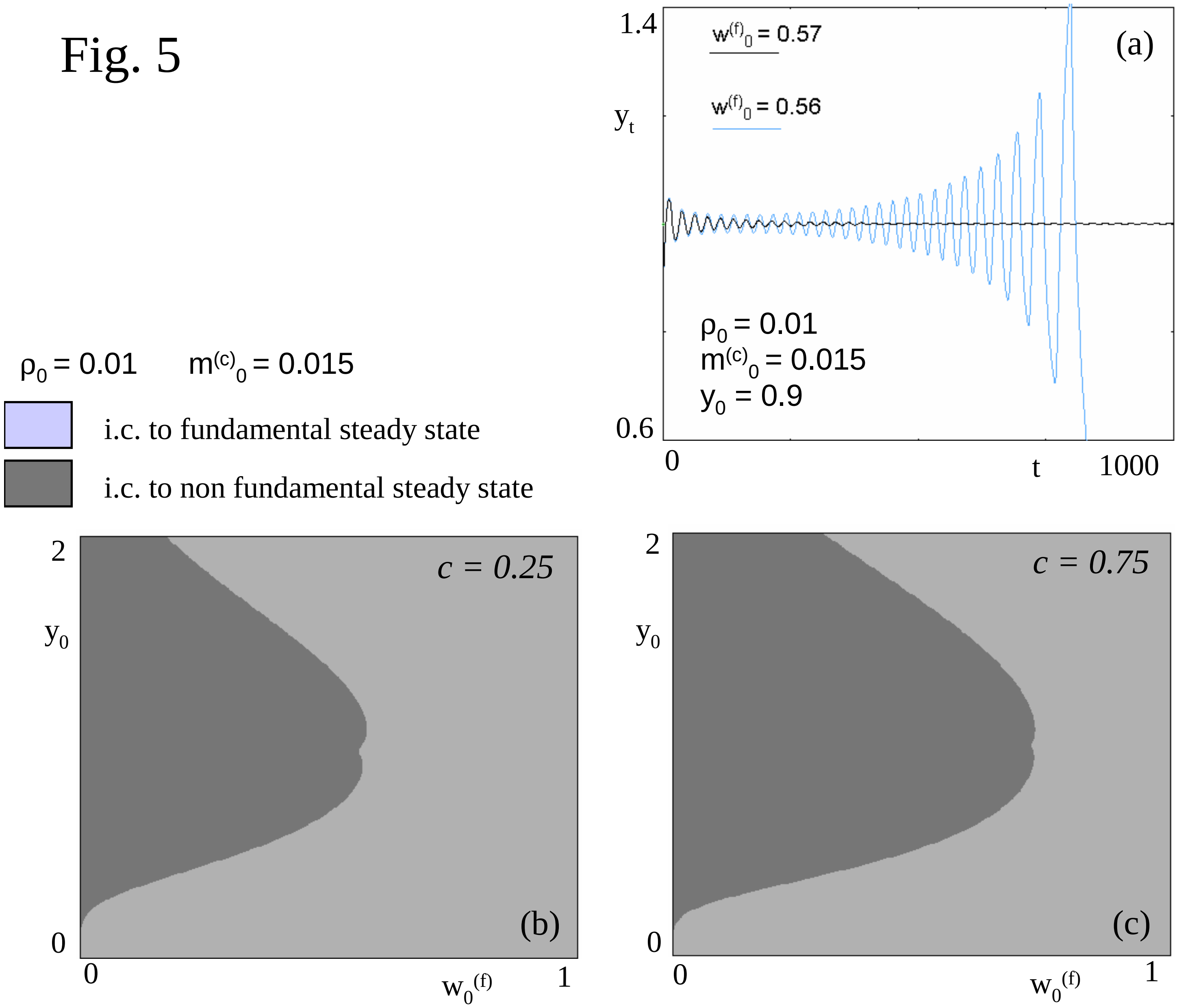
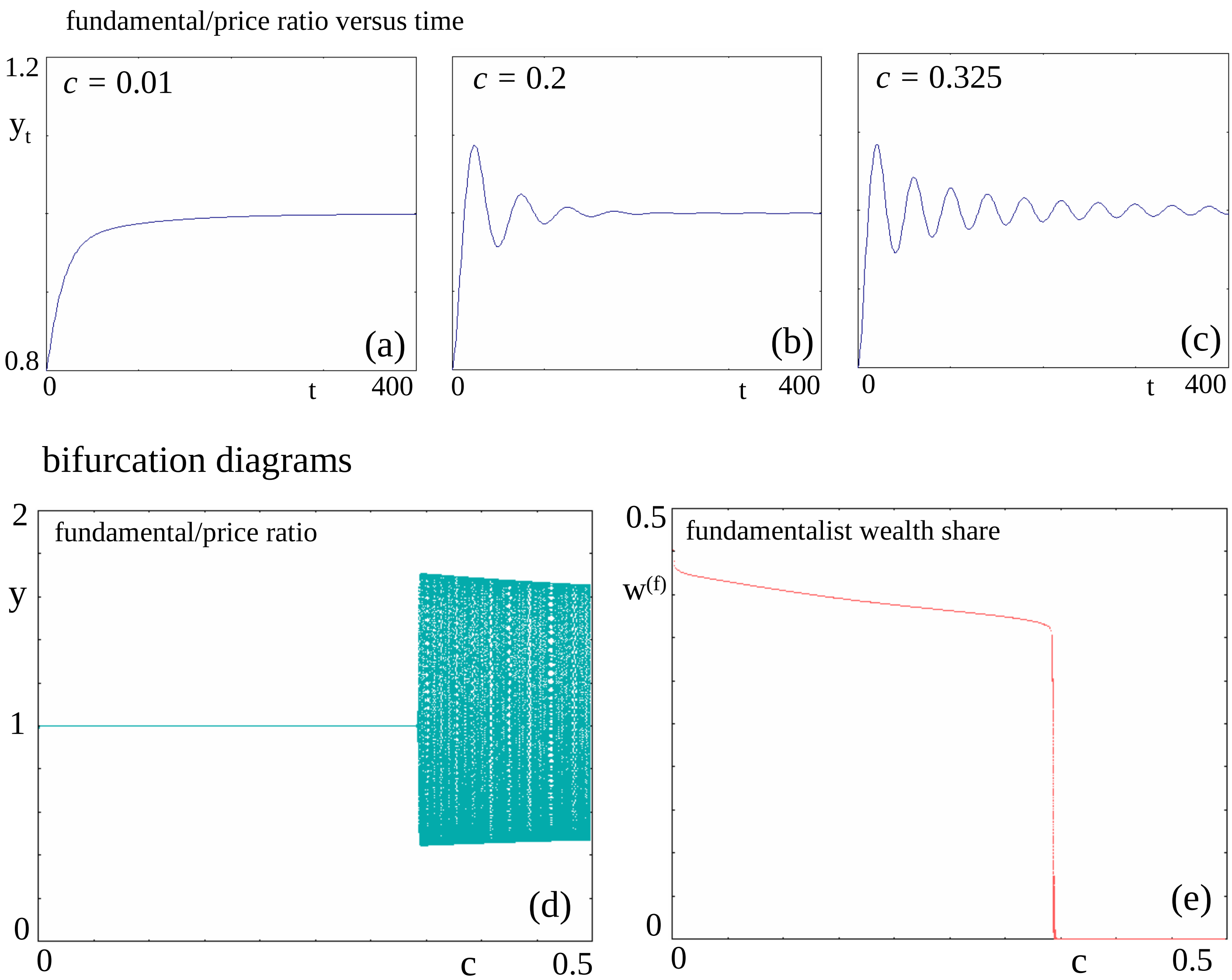
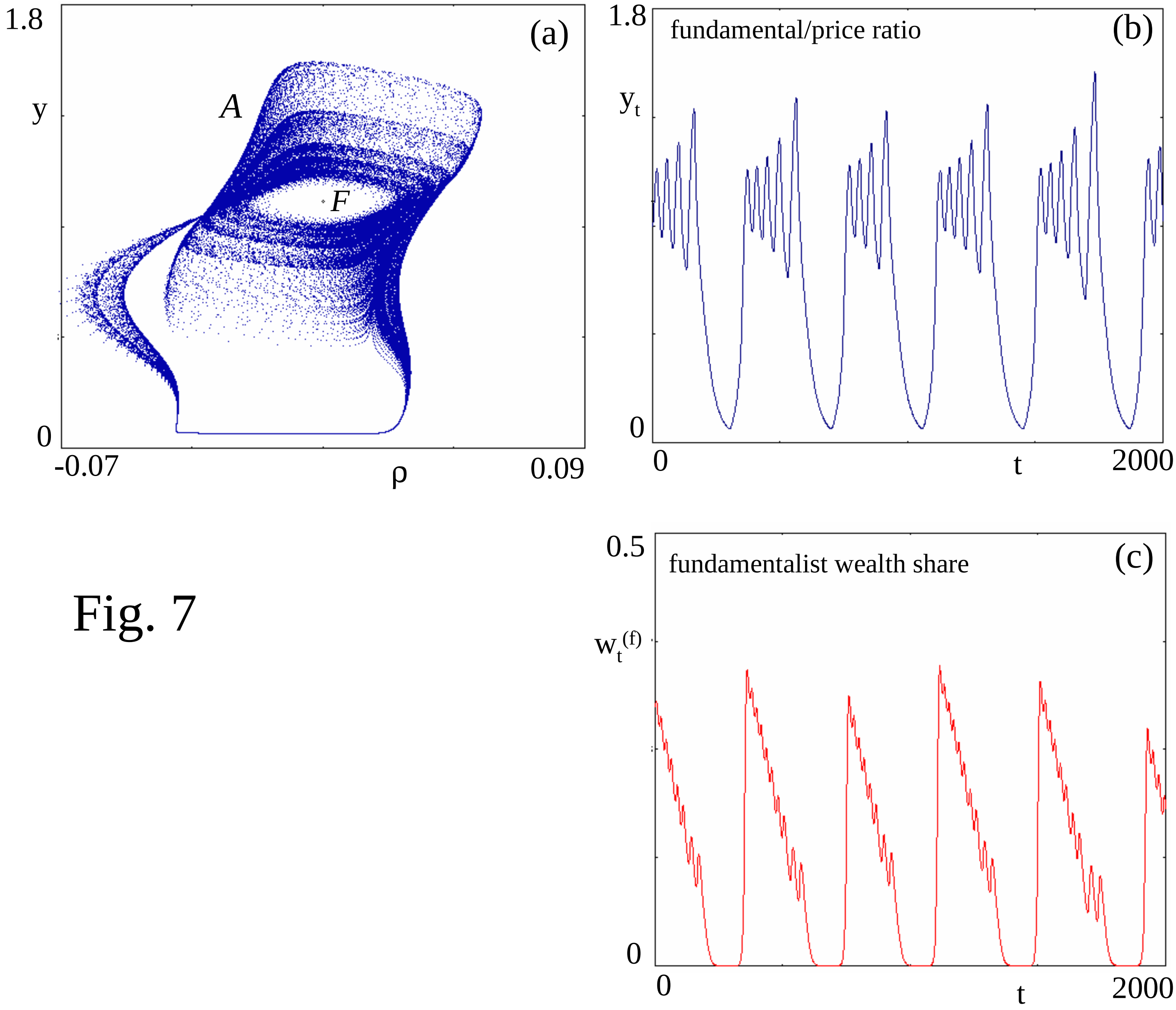

1.4
$w^{(f)}_0 = 0.57$
$w^{(f)}_0 = 0.56$
(a)
y_t
$\rho_0 = 0.01$
$m^{(c)}_0 = 0.015$
$y_0 = 0.9$
0.6
0
t
1000
$\rho_0 = 0.01$ $m^{(c)}_0 = 0.015$
i.c. to fundamental steady state
i.c. to non fundamental steady state
2
y_0
c = 0.25
(b)
0
0
$w_0^{(f)}$
1
2
y_0
c = 0.75
(c)
0
0
$w_0^{(f)}$
1

Fig. 6

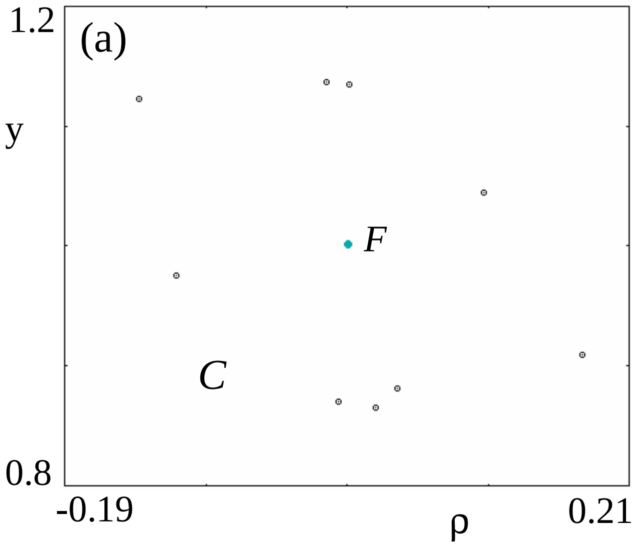

fundamental/price ratio versus time
1.2
y_t
0.8
c = 0.01
(a)
0
t
400
c = 0.2
(b)
0
t
400
c = 0.325
(c)
0
t
400
bifurcation diagrams
2
y
1
0
fundamental/price ratio
(d)
0
c
0.5
0.5
$w^{(f)}$
0
fundamentalist wealth share
(e)
0
c
0.5



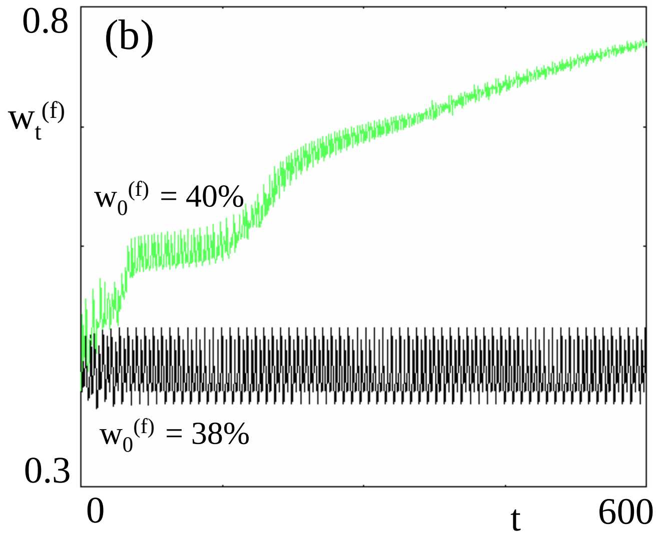
1.8
y
A
F
0
-0.07
ρ
0.09
(a)
1.8
fundamental/price ratio
(b)
y_t
0
0
t
2000
0.5
fundamentalist wealth share
(c)
w_t^(f)
0
0
t
2000

Fig. 7

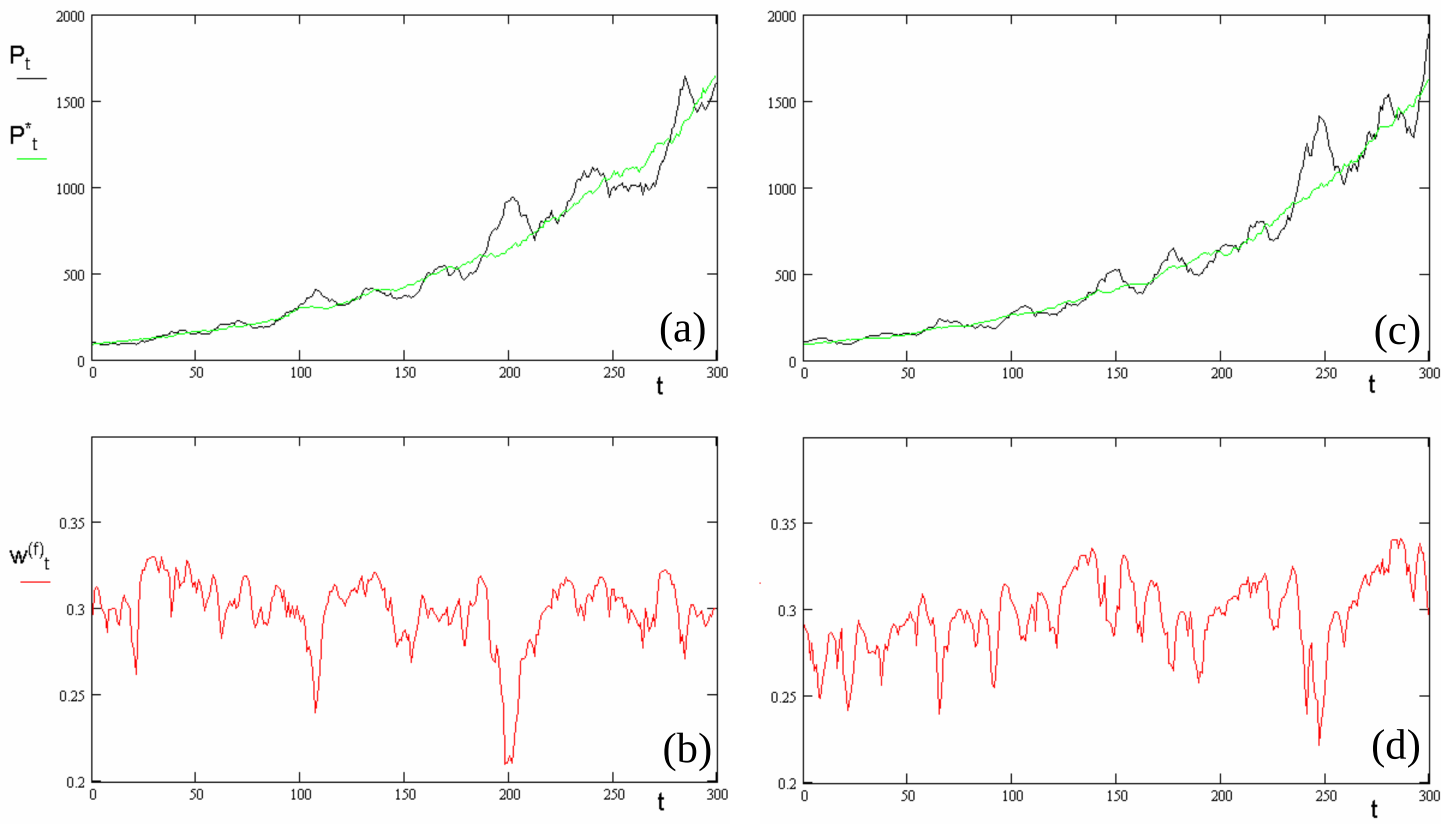
Fig. 8



(a)



(b)


P_t
P*_t
w^(f)_t
t
(a)
(b)
(c)
(d)
2000
1500
1000
500
0.35
0.3
0.25
0.2
0
50
100
150
200
250
300

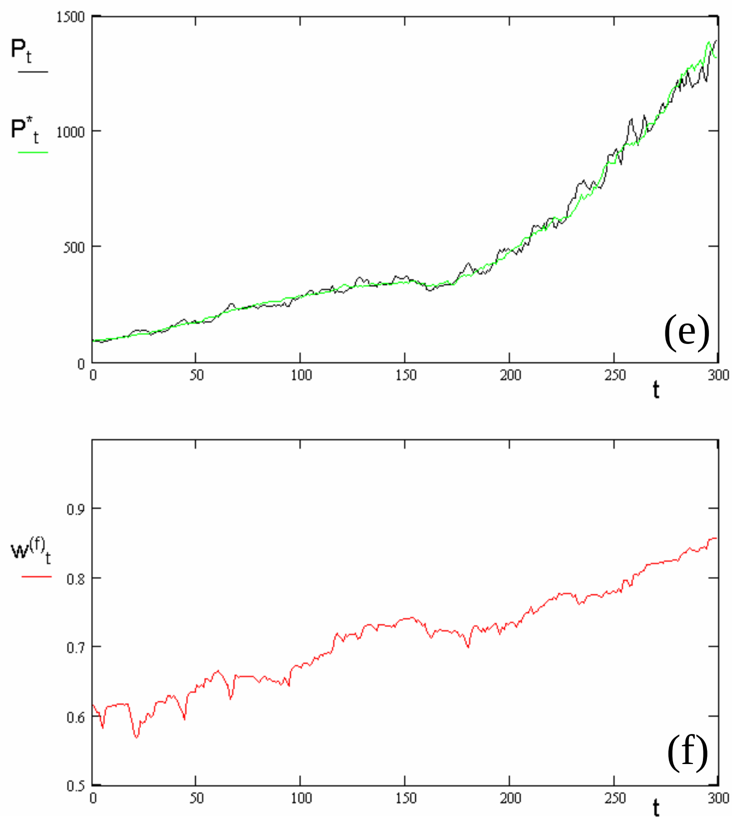

P_t
P*_t
w^(f)_t
t
(e)
(f)
1500
1000
500
0.9
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300

Fig. 9