REDUCTION, EMULATION, AND EMERGENCE IN DYNAMICAL SYSTEMS[†] [‡]

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ABSTRACT

The received view about emergence and reduction is that they are incompatible categories. I argue in this paper that, contrary to the received view, emergence and reduction can hold together. To support this thesis, I focus attention on dynamical systems and, on the basis of a general representation theorem, I argue that, as far as these systems are concerned, the emulation relationship is sufficient for reduction (intuitively, a dynamical system DS_1 emulates a second dynamical system DS_2 when DS_1 exactly reproduces the whole dynamics of DS_2). This *representational* view of reduction, contrary to the customary *deductivist* one, is compatible with the existence of structural properties of the reduced system that are not also properties of the reducing one. According to this view, reduction is better analyzed in terms of a representational relationship between *models*, rather than a deductive relationship between *theories*. Reduction and emergence may co-occur, for they are complementary manifestations of an underlying representational relationship between mathematical models, namely, the one of *emulation*.

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1 Introduction

Emergence and reduction are traditionally viewed as incompatible categories (Beckermann 1992; Kim 1992). A property of a high level system is said to be emergent if it cannot be explained in terms of properties of the system's constitutive entities or, in other words, if it is not one of the properties of the basic building blocks that, together, make up the system. Thus, in order to speak of an emergent property P of system S_2 we need to verify, first, that S_2 is made up of a different system S_1 (intuitively, S_1 is the system of the constitutive entities of S_2 taken in isolation, or in relations different from those typical of S_2) and, second, that P is not one of the properties of S_1. But then, the concept of emergence seems to yield a paradox: On the one hand, since S_2 *is made up of* S_1, S_2 is reduced to S_1; on the other one, since the property P of S_2 *is not one of the properties of* S_1, S_2 is not reduced to S_1. The traditional solution denies that the constitution relationship (S_2's being made up of S_1) is sufficient for reduction. By contrast, the second horn of the dilemma is not questioned, for it is taken for granted that S_2's reduction to S_1 entails that any property of S_2 is also a property of S_1.

This paper maintains that the traditional solution is flawed. In fact, there are pairs of systems, S_2 and S_1, for which both the constitution relationship (S_2 is made up of S_1) and the reduction one (S_2 is reduced to S_1) clearly hold together. Moreover, for these pairs of systems, it also turns out that some property of S_2 is not a property of S_1, so that any such property is emergent. It follows that, contrary to the received view, emergence and reduction are by no means incompatible categories but, rather, complementary ones.

To support this thesis, I will consider some simple examples of dynamical systems for which the emulation relationship holds. As intended here (Arnold 1977; Szlensk 1984; Giunti 1997), a *dynamical system* is a kind of mathematical model that captures the intuitive idea of an arbitrary deterministic system. Models of this kind allow us to study in a precise way typical features of complex systems. Among them, in recent years, the one of emulation has gained growing attention (Wolfram 1983a, 1983b, 1984a, 1984b, 2002). Intuitively, a dynamical system DS_1 *emulates* a second dynamical system DS_2 when the first one exactly reproduces the whole dynamics of the second one.

The emulation relationship can be defined in a precise way for any two arbitrary dynamical systems and it has been shown (Giunti 1997, ch.1, th. 11) that, if DS_1 emulates DS_2, there is a third system DS_3 such that (i) DS_2 is isomorphic to DS_3; (ii) all states of DS_3 are states of DS_1; (iii) any state transition of DS_3 is constructed out of state transitions of DS_1. In this paper, I will prove a more general version of this theorem [*Virtual System Theorem VST*], which is based on a weaker and simpler definition of emulation. I will then argue that this result allows us to claim: If DS_1 emulates DS_2, then DS_2 is *made up* of DS_1 and DS_2 is *reduced* to DS_1. Therefore, to show that both reduction and emergence can hold together, it will suffice to exhibit two dynamical systems DS_1 and DS_2, and a property P, such that DS_1 emulates DS_2, DS_2 has P, but DS_1 does not have P. I will show that this situation already obtains for two pairs of simple finite discrete systems and that, in either case, the emergent property P is a strong form of irreversibility of system DS_2.

The claim that emulation is sufficient for both constitution and reduction (in force of [*VST*]) is the basis for a unified *representational* view of reduction and emergence. According to it, reduction is better analyzed in terms of a representational relationship between models (i.e. emulation) rather than a deductive relationship between theories. Strictly speaking, this claim is intended to hold exclusively for dynamical systems as purely *mathematical models* with no empirical interpretation. In a different sense, however, dynamical systems typically function as *models of real phenomena*. In this second sense, a dynamical system is not a purely mathematical entity DS, but it is a pair (DS, I_H), where I_H is an empirical interpretation that links the purely mathematical model DS to a phenomenon H. Sec. 7 will provide the main lines of an extension of the representational theory of reduction to empirically interpreted dynamical systems.

As said, the emulation relationship is the basis of a unified approach to reduction and emergence. The simplest form of such relationship holds between two dynamical systems DS_1 and DS_2 when the *whole* dynamics of DS_2 is *exactly* reproduced by DS_1. This simple form may very well be the basis for a representational account of *total* and *exact* reduction, but we need a more sophisticated version of emulation for dealing with cases of asymptotic, partial and approximate reduction (Hooker 2004). Such a version will be introduced in the Appendix, where it will then be employed for a treatment of partial and approximate reduction in empirically interpreted dynamical systems.

2 Dynamical systems and emulation

A dynamical system is a kind of mathematical model that formally expresses the notion of an arbitrary deterministic system, either reversible or irreversible, with discrete or continuous time or state space. Let Z be the integers, Z^+ the non-negative integers, R the reals and R^+ the non-negative reals; below is the exact definition of a dynamical system.

[1] *DS is a dynamical system* iff there is M, T, $(g^t)_{t \in T}$ such that $DS = (M, (g^t)_{t \in T})$ and

1. M is a non-empty set; M represents all the possible states of the system, and it is called the *state space*;
2. T is either Z, Z^+, R, or R^+; T represents the time of the system, and it is called the *time set*;

3. $(g^t)_{t \in T}$ is a family of functions from M to M; each function g^t is called a *state transition* or a *t-advance* of the system;
4. for any $t, v \in T$, for any $x \in M$, $g^0(x) = x$ and $g^{t+v}(x) = g^v(g^t(x))$.

[2] A *discrete dynamical system* is a dynamical system whose state space is finite or denumerable, and whose time set is either Z or Z^+; examples of discrete dynamical systems are Turing machines and cellular automata.[1] [3] A *continuous dynamical system* is a dynamical system that is not discrete; examples of continuous dynamical systems are iterated mappings on R, and systems specified by ordinary differential equations.

[4] $DS = (M, (g^t)_{t \in T})$ is a *possible dynamical system* iff DS satisfies the first three conditions of definition [1]. We can now define the concept of an isomorphism between two possible dynamical systems as follows. [5] *r is an isomorphism of* DS_1 *in* DS_2 iff $DS_1 = (M, (g^t)_{t \in T})$ and $DS_2 = (N, (h^v)_{v \in V})$ are possible dynamical systems, $T = V$, $r: M \to N$ is a bijection and, for any $t \in T$, for any $x \in M$, $r(g^t(x)) = h^t(r(x))$.

[6] DS_1 *is isomorphic* to DS_2 iff there is r such that r is an isomorphism of DS_1 in DS_2. It is easy to verify that the isomorphism relation is an equivalence relation on any given set of possible dynamical systems. (The concept of *set of all possible dynamical systems* is inconsistent, and we must then take as the basis of the theory of dynamical systems a specific, sufficiently large, set of possible dynamical systems.)

It is also not difficult to prove that the relation of isomorphism is compatible with the property of being a dynamical system, that is to say: if DS_1 is isomorphic to DS_2 and DS_1 is a dynamical system, then DS_2 is a dynamical system. This allows us to speak of abstract dynamical systems in exactly the same sense we talk of abstract groups, fields, lattices, order structures, etc. We can thus define: [7] an *abstract dynamical system* is any equivalence class of isomorphic dynamical systems. It is easily shown that any two dynamical systems have exactly the same structural properties iff they are isomorphic.[2] Since general dynamical systems theory[3] is exclusively interested in such properties, it regards any two isomorphic systems as identical.

Dynamical systems are appropriate models to study several interesting features of complex systems. The one of emulation is typical of computational systems (Wolfram 2002), but it can in principle involve any two dynamical systems. The intuitive idea is that a dynamical system DS_1 emulates a second dynamical system DS_2 when the first one exactly reproduces the whole dynamics of the second one. Here are some examples. A universal Turing machine emulates any Turing machine; for any Turing machine *TM* there is a cellular automaton *CA* such that *CA* emulates *TM* (Smith 1971, th. 3), and vice versa; the simple cellular automaton specified by Wolfram's rule 18 emulates the one specified by rule 90 (both *CA* are monodimensional, with 2 possible values for cell, and neighborhood of radius 1; see Wolfram 1983b, 20).

Giunti 1997 (ch. 1, def. 4) gave a formal definition of the emulation relationship that applies to any two arbitrary dynamical systems. Here, I will employ a weaker and simpler definition (see figure 1), which nevertheless suffices for the present purposes.

[8] DS_1 *emulates* DS_2 iff $DS_1 = (M, (g^t)_{t \in T})$ and $DS_2 = (N, (h^v)_{v \in V})$ are dynamical systems, and there is an injective function $u: N \to M$ such that, for any $v \in V$, for any $c \in N$, there is $t \in T$ such that $u(h^v(c)) = g^t(u(c))$. Any function u that satisfies the previous condition is called an *emulation of* DS_2 *in* DS_1.

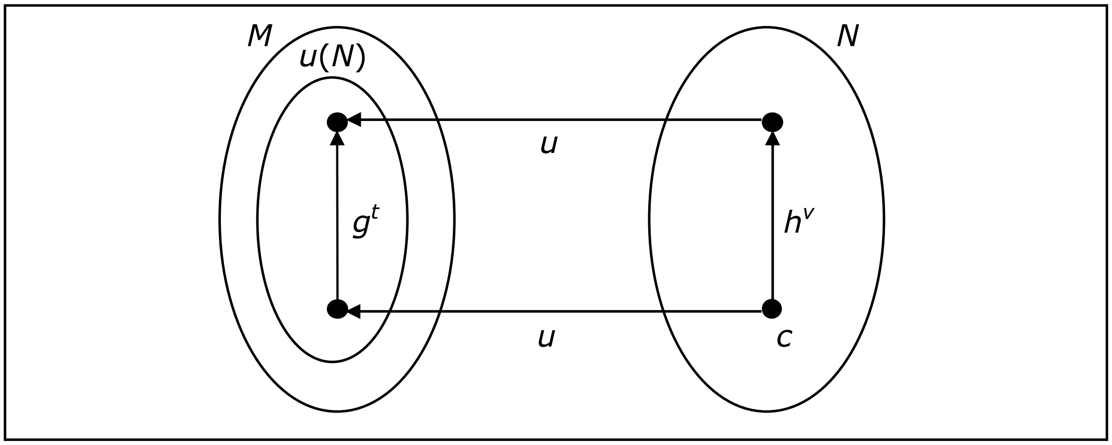

FIGURE 1 Emulation

3 Emulation, constitution, and reduction

Giunti 1997 (ch. 1, th. 11) proved that, if u is an emulation of DS_2 in DS_1, there is a third system DS_3 such that (i) u is an isomorphism of DS_2 in DS_3; (ii) all states of DS_3 are states of DS_1; (iii) any state transition of DS_3 is constructed out of state transitions of DS_1. This result still holds for the weaker definition of emulation [8], as the following theorem shows.

Virtual System Theorem [*VST*]

- Let $DS_1 = (M, (g^t)_{t \in T})$ and $DS_2 = (N, (h^v)_{v \in V})$ be dynamical systems, and u be an emulation of DS_2 *in* DS_1;
- let $DS_3 = (\underline{N}, (\underline{h}^v)_{v \in V})$, where $\underline{N} = u(N)$ and, for any $a \in \underline{N}$, for any $v \in V$, $\underline{h}^v(a) = u(h^v(u^{-1}(a))$; the system DS_3 is called *the u-virtual system* DS_2 *in* DS_1 (see figure 2);

then:

(i) u is an isomorphism of DS_2 in DS_3;
(ii) all states of DS_3 are states of DS_1;
(iii) for any state transition $\underline{h}^v$ of DS_3, for any $a \in \underline{N}$, there is a state transition g^t of DS_1 such that $\underline{h}^v(a) = g^t(a)$.

Proof of (i)
By the definition of DS_3, for any $c \in N$, $u(h^v(c)) = u(h^v(u^{-1}(u(c))) = \underline{h}^v(u(c))$. Therefore, by the definition of isomorphism [5], u is an isomorphism of DS_2 in DS_3.

Proof of (ii)
Obvious, by the definition of DS_3.

Proof of (iii)
By the definition of DS_3, for any $v \in V$, for any $a \in \underline{N}$, $\underline{h}^v(a) = u(h^v(u^{-1}(a))$. Let $c = u^{-1}(a)$. Since u is an emulation of DS_2 in DS_1, by definition [8], there is $t \in T$ such that $u(h^v(c)) = g^t(u(c))$. Therefore, $\underline{h}^v(a) = g^t(u(c)) = g^t(a)$. Q.E.D.

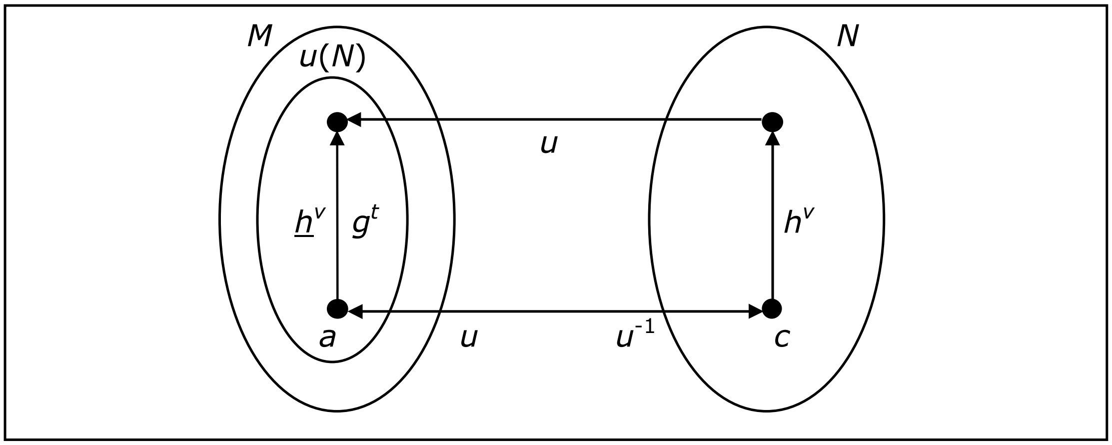

FIGURE 2 The u-virtual system DS_2 in DS_1

It is my contention that, if a dynamical system DS_1 emulates a second system DS_2, $[VST]$ allows us to claim that DS_2 *is made up of* DS_1, as well as that DS_2 *is reduced to* DS_1. In other words, I maintain that, because of $[VST]$, emulation[4] is sufficient for both *constitution* and *reduction*.

Before seeing the details of the supporting argument, it is important to make clear that dynamical systems, as intended here, are *purely mathematical* entities with no empirical interpretation; that is to say, at this level of analysis, a dynamical system is just a model of the mathematical theory whose Suppes' style axiomatization (1957, ch. 12) is given by def. [1]. The claim that emulation is sufficient for constitution and reduction is thus exclusively limited to dynamical systems intended in this sense.

As just said, when I speak of a dynamical system as a *model*, I mean a model of a quite general *mathematical theory*, whose axiomatization is expressed by the definition, in set theory, of an appropriate set-theoretical predicate (def. [1]). It is important to sharply distinguish this sense of the term "model" from a different one, which also applies to dynamical systems, and is equally central to a complete understanding of their epistemological status. This second sense is the one intended when we say that a specific dynamical system is a model of a *real phenomenon*; however, this sense does not refer to a dynamical system as a purely mathematical entity (i.e., just a model of general dynamical system theory) but, rather, to such entity *together with* an empirical interpretation that links the mathematical model to the phenomenon which it is intended to describe.

A simple example will make the distinction clear. Let us consider the following system of two ordinary differential equations $\langle dx(v)/dv = \boldsymbol{k},\ dy(v)/dv = x(v)\rangle$, where $\boldsymbol{k}$ is a fixed real positive constant. The solutions of such equations uniquely determine the dynamical system $DS_e = (X{\times}Y, (h^v)_{v\in V})$, where $X = Y = V = R$ (the real numbers) and, for any $v, x, y \in R$, $h^v(x, y) = (\boldsymbol{k}v + x,\ \boldsymbol{k}v^2/2 + xv + y)$. It is immediate to verify that DS_e satisfies def. [1], so that it is a model in the first sense. On the other hand, let us consider the phenomenon of the free fall of a medium size body in the vicinity of the earth (henceforth, H_e), and let us interpret the first component X of the state space of DS_e as the set of all possible values of the *vertical velocity* of an arbitrary free falling body, the second component Y as the set of all possible values of the *vertical position* of the falling body, and the time set V of DS_e as the set all possible instants of *physical time*. Since all three of these magnitudes are measurable or detectable properties of the intended phenomenon H_e, the given interpretation is an *empirical* interpretation of the dynamical system DS_e on H_e. Let I_{H_e} be such an interpretation. Then, the pair (DS_e, I_{H_e}) is a model of H_e, that is to say, such a *pair* is a model in the second sense.[5]

My claim that emulation is sufficient for both constitution and reduction (in force of $[VST]$) is intended to hold exclusively for dynamical models in the first sense. This does not mean that such a claim does not have any bearing on the further question: What are the conditions for reduction of an empirically interpreted dynamical system (DS_2, I_2) to another one (DS_1, I_1)? I will return later (see sec. 7) to this question. For the moment, it suffice to say that, in my view, the conditions for reduction of the mathematical model DS_2 to the mathematical model DS_1 are a necessary component of the more complex conditions for reduction of (DS_2, I_2) to (DS_1, I_1).

I am now going to present a detailed argument to support the claim that emulation is sufficient for both constitution and reduction. For the sake of clarity, the argument is divided in two distinct sub-arguments, the first one for constitution, and the second one for reduction. The complete argument relies on six premises, divided into three groups. The premises of the first group (**A1** and **A2**) are the most general ones, for they refer to systems of *any* kind. Specifically, **A1** states a sufficient condition for constitution, and **A2** a sufficient condition for reduction between two arbitrary systems. The premises of the second group (**B1** and **B2**) are at an itermediate level of generality, for they refer exclusively to *mathematical* systems of any kind, that is, systems that are models of *some* mathematical theory. **B1** explicitly states what it is to be intended for "constitutive entity of a mathematical model", while **B2** makes clear the meaning of "whole structure of a mathematical model". The premises of the third group (**C1** and **C2**) are the most specific, for they refer to *dynamical systems* (in the purely mathematical sense). In particular, **C1** states identity conditions for such systems, and **C2** makes explicit the exact meaning of "whole structure of a dynamical system". Below are the six premises. Each of them is followed by a brief elucidation, which is intended to pin point crucial features of the corresponding premise, as well as to provide an intuitive justification for its assumption.

A1 For a system S_2 to be made up (or constituted) of a system S_1, it is sufficient that all the constitutive entities of S_2 are constitutive entities of S_1. *Elucidation* — Let us consider a system S_1 all of whose structural elements are constructed by means of a given stock of building blocks; let us call such building blocks "the constitutive entitities of S_1". If we then construct a second system S_2 by only employing constitutive entities of S_1, we say that S_2 is made up (or constituted) of S_1.

A2 For a system S_2 to be reduced to a system S_1, it is sufficient that (a) all the constitutive entities of S_2 are constitutive entities of S_1 and (b) the whole structure of S_2 is a part of the whole structure of S_1. *Elucidation* — In general, a system S is characterized by a whole structure formed by a complex of interconnected elements; each of these structural elements is built out of a given stock of building blocks, which we call "the constitutive entities of S". Thus, if two systems S_1 and S_2 satisfy conditions (a) and (b) above, the system S_2 is in fact a subsystem of S_1; this allows us to claim that S_2 is reduced to S_1.

B1 The constitutive entities of a mathematical model are the entities in its domain. *Elucidation* — According to standard definition, a mathematical model *MS* is a set *D* together with a family $(\sigma_i)_{i\in I}$ of relations on *D*. For any $i \in I$, there is exactly one $n \geq 0$ such that σ_i has arity *n*, where relations of arity 0 are identified with members of *D*, and relations of arity $n > 0$ are identified with sets of *n*-tuples of members of *D*; the set *D* is called the *domain* of the model. A mathematical model can thus be thought as a special kind of system, whose structural elements are the relations in the family $(\sigma_i)_{i\in I}$, and whose constitutive entities are the members of *D*.

B2 The *whole* structure of a mathematical model $MS = (D, (\sigma_i)_{i\in I})$ is the union of all the relations in the family $(\sigma_i)_{i\in I}$;[6] accordingly, if the relata of "is a part of" are whole structures of mathematical models, "is a part of" is to be interpreted as set-inclusion. *Elucidation* — We have just seen that a mathematical model can be thought as a special kind of system, whose structural elements are the relations in the family $(\sigma_i)_{i\in I}$. Each of such relations is a set of *n*-tuples; thus, the union of these sets is the whole structure formed by the complex of such relations. Given this interpretation of "whole structure of a mathematical model", it is then obvious that "is a part of" should be interpreted as set-inclusion.

C1 From the point of view of general dynamical systems theory, any two isomorphic dynamical systems are identical. *Elucidation* — General dynamical systems theory studies the structural properties (see notes 2 and 3) of dynamical systems, and any two dynamical systems have exactly the same structural properties iff they are isomorphic. Therefore, general dynamical systems theory does not distinguish between any two isomorphic dynamical systems.

C2 If a mathematical model is a dynamical system $DS = (M, (g^t)_{t\in T})$, the whole structure of the model is the set of all state pairs (x, y) such that, for some $t \in T$, $g^t(x) = y$. *Elucidation* — We should first of all notice that, by def. [1], a dynamical system is a mathematical model of a special kind, namely, such that any relation g^t is in fact a function from *M* to *M*. Then, **C2** is an immediate consequence of this observation and **B2**.[7]

SUFFICIENCY OF EMULATION FOR CONSTITUTION

1. For a mathematical model MS_2 to be made up of a mathematical model MS_1, it is sufficient that the domain of MS_2 is included in the domain of MS_1; (logically follows from **A1** and **B1**)
2. hence, if *u* is an emulation of DS_2 in DS_1, the *u*-virtual system DS_2 in DS_1 is made up of DS_1; (logically follows from 1 and thesis (ii) of [*VST*])
3. if *u* is an emulation of DS_2 in DS_1, DS_2 is isomorphic to the *u*-virtual system DS_2 in DS_1; (logically follows from thesis (i) of [*VST*] and def. [6])
4. consequently, if *u* is an emulation of DS_2 in DS_1, DS_2 is made up of DS_1. (Logically follows from 2, 3, **C1** and the fact that dynamical systems, as intended here, are just models of general dynamical systems theory)

SUFFICIENCY OF EMULATION FOR REDUCTION

1. For a mathematical model MS_2 to be reduced to a mathematical model MS_1, it is sufficient that (a) the domain of MS_2 is included in the domain of MS_1 and (b) the whole structure of MS_2 is included in the whole structure of MS_1; (logically follows from **A2**, **B1** and **B2**)

2. hence, if u is an emulation of DS_2 in DS_1, the u-virtual system DS_2 in DS_1 is reduced to DS_1; (logically follows from 1, **C2**, and theses (ii) and (iii) of $[VST]$)
3. if u is an emulation of DS_2 in DS_1, DS_2 is isomorphic to the u-virtual system DS_2 in DS_1; (logically follows from thesis (i) of $[VST]$ and def. [6])
4. consequently, if u is an emulation of DS_2 in DS_1, DS_2 is reduced to DS_1. (Logically follows from 2, 3, **C1** and the fact that dynamical systems, as intended here, are just models of general dynamical systems theory)

4 Emergence and reduction

P is an emergent property of system S_2 with respect to system S_1 just in case (a) S_2 is made up of S_1 (intuitively, S_1 is the system of the constitutive entities of S_2 taken in isolation, or in relations different from those typical of S_2) and (b) P is a property of S_2 but P is not one of the properties of S_1.

The previous characterization of an emergent property is not a formal definition; it is rather an explicit formulation of one of the senses of the term "emergence" that is quite standard in the systems science literature (Holland 1998, 121-122; Minati 2004, sec. 2.2). In the philosophical camp, a closely related sense was proposed by Broad (1925, ch. 2; also see Beckermann 1992, sec. 2); the following characterization is intended to express the basic insights of Broad's view of emergence.

P is an emergent property of system S_2 just in case there is a system S_1 (intuitively, S_1 is the system of the constitutive entities of S_2 taken in isolation, or in relations different from those typical of S_2) such that S_2 is made up of S_1, P is a property of S_2 but P is not a property of S_1 and, for any other system $S_1^* \neq S_2$, if S_2 is made up of S_1^*, P is not a property of S_1^*.

It thus follows that P is an emergent property of system S_2 iff for *any* system S_1 such that $S_1 \neq S_2$ and S_2 is made up of S_1, P is an emergent property of S_2 with respect to S_1.[8] Therefore, Broad's notion of emergence is in fact an *absolute* version of the system's science *relative* notion. As Broad himself explicitly remarked (1925, ch. 2), it is quite difficult to give *conclusive* examples of the absolute notion of emergence, for we should somehow be able to exclude the existence of *any* system $S_1^* \neq S_2$ such that S_2 is made up of S_1^* and P is a property of S_1^*, where P is a given property of system S_2.

This problem does not arise when the relative notion of emergence is adopted. For a given system S_2, once a reference system S_1 and a property P are appropriately chosen, it is not usually very hard to show that (a) S_2 is made up of S_1 and (b) P is a property of S_2 but P is not one of the properties of S_1. In systems science, a typical choice of the reference system consists in identifying S_1 with the system of all the constitutive entities of S_2 taken *in isolation* (henceforth, the *default* choice).[9] With the default choice of S_1, the claim that S_2 has emergent properties (in the relative sense) reduces to the familiar phrase:

> The behavior of the overall system [all the properties of system S_2] cannot be obtained by summing the behaviors [properties] of its constituent parts [system S_1]. (Holland 1998, 122)

As regards the relationship between emergence and reduction, in systems science we can distinguish at least two opposing views. On the one hand, reduction is often identified with the possibility of obtaining the behavior of a complex system by summing the behaviors of its constituent parts, taken in isolation. Thus, on this view, reduction is just the opposite of emergence (where the choice of the reference system S_1 is the default one). On the other hand, reduction is viewed as the possibility of explaining the behavior of a complex system in terms of the behaviors of its constituent parts; however, such parts are *not* taken in isolation, but they are considered within the specific pattern of *interaction* proper of the complex system itself (see Holland 1998, 13-14; 122). On this second view, not only do emergence and reduction occur together, but in fact emergence even presupposes reduction:

> The program for studying emergence set forth here depends on *reduction*. Complicated systems are described in terms of *interactions* of simpler systems … I emphasize "interactions" because there is a common misconception about reduction: to understand the whole, you analyze the process into atomic parts, and then study these parts in isolation. Such analysis works when the whole can be treated as the *sum* of its parts, but it does *not* work when the parts interact in less simple ways … Emergence in the sense used here occurs only when the activities of the parts do *not* simply sum to give activity of the whole. For emergence, the whole is indeed more than the sum of its parts. (Holland 1998, 13-14)

It should be noticed that, on Holland's view, both reduction and emergence are relations between a property and a system, and emergence and reduction co-occur because the same property turns out to be reduced with respect to one system and emergent with respect to a *different* one. In fact, when we consider a complex system S_2 and one of its properties P, P is reduced to S_2 if it is explained in terms of the specific pattern of interaction characteristic of S_2 itself; but, on the other hand, P turns out to be emergent with respect to the default reference system S_1, that is, the system of the constituent parts of S_2 taken in isolation. Thus, according to Holland, the term "reduction" is in fact a synonym for "explanation of a property in terms of the specific interaction laws of the system that has such property". It should also be noticed that Holland's usage is not standard, for reduction is normally intended as a relation between two different *systems*.

In what follows, I will assume the relative notion of emergence, and I will exhibit two cases in which emergence and reduction co-occur. These cases, unlike the ones considered by Holland, are such that (i) reduction is a relationship between two different systems, and not between a property and the system that has such property; (ii) emergence and reduction co-occur with respect to the *same* reference system.

Since emulation is sufficient for both constitution and reduction, in order to show that emergence and reduction simultaneously occur with respect to the same system, it is sufficient to exhibit a pair of dynamical systems DS_1 and DS_2, as well as a property P, such that DS_1 emulates DS_2, DS_2 has P, but DS_1 does not have P. In the next section, I will give two examples of such pairs of systems. For each pair, both DS_1 and DS_2 are small finite discrete systems (with just three states), while the emergent property P is the strong irreversibility[10] of system DS_2.

5 Examples of dynamical systems DS_1 and DS_2 such that (i) DS_2 is reduced to DS_1 and (ii) DS_2 has emergent properties with respect to DS_1

To state the examples, we first need a few more concepts of general dynamical systems theory. [9] A *cascade* is a dynamical system with discrete time, i.e., whose time set is either Z or Z^+. [10] A dynamical system is *reversible* iff its time set is either Z or R; [11] it is *irreversible* iff its time set is either Z^+ or R^+. Note that any t-advance g^t ($t > 0$) of an irreversible cascade $(M, (g^t)_{t\in Z^+})$ can always be thought as being generated by iterating t times a given function $g: M \to M$ (where $g = g^1$).[11] Therefore, as far as an irreversible cascade is concerned, the whole dynamics of the system is entailed by the behavior of its first t-advance g^1.

[12] A dynamical system is *logically reversible* iff it is irreversible, but all its state-transitions are injective; [13] it is *logically irreversible* iff it is irreversible and at least one of its state-transitions is not injective; [14] it is *strongly irreversible* iff there are two different states a and b and a state-transition g^v such that $g^v(a) = g^v(b)$ and, for any state-transition g^t, $g^t(a) \neq b$ and $g^t(b) \neq a$. Obviously, by definitions [12], [13] and [14], if a dynamical system is logically reversible, it is neither logically irreversible nor strongly irreversible.[12]

Figure 3 shows a pair of cascades $DS_1 = (M, (g^t)_{t\in Z^+})$ and $DS_2 = (N, (h^v)_{v\in Z^+})$. The state space of DS_1 is $M = \{x, y, z\}$, and that of DS_2 is $N = \{a, b, c\}$. Each state-transition g^t of DS_1 is obtained by applying t-times the state-transition g^1, defined by: $g^1(x) = y$, $g^1(y) = z$, $g^1(z) = z$; analogously, an arbitrary state-transition h^v of DS_2 is obtained from the first state-transition h^1, defined by: $h^1(a) = c$, $h^1(b) = c$, $h^1(c) = c$. The function $u: N \to M$ is defined as follows: $u(a) = x$, $u(b) = y$, $u(c) = z$. Figure 3 then shows that (a) u is an emulation of DS_2 is DS_1, (b) DS_1 is logically irreversible but not strongly irreversible, (c) DS_2 is strongly irreversible. From this, since emulation is sufficient for both constitution and reduction, it follows that (i) DS_2 is reduced to DS_1 and (ii) the property P of strong irreversibility is an emergent property of DS_2 with respect to DS_1.

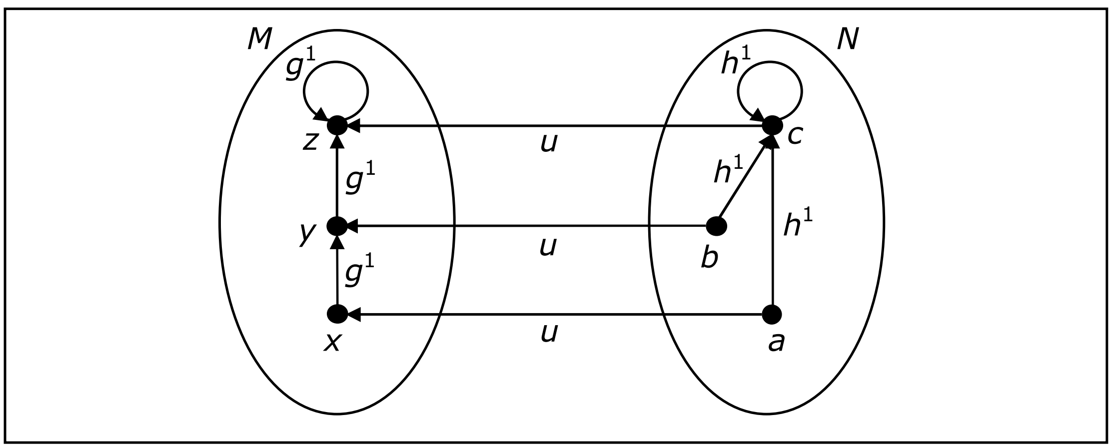

FIGURE 3 DS_1 emulates DS_2, DS_1 is logically irreversible but not strongly irreversible, and DS_2 is strongly irreversible

Figure 4 shows a second pair of cascades $DS_1 = (M, (g^t)_{t\in Z^+})$ and $DS_2 = (N, (h^v)_{v\in Z^+})$ such that (i) DS_2 is reduced to DS_1 and (ii) the property P of strong irreversibility is an emergent property of DS_2 with respect to DS_1. Note that DS_2 is identical to the corresponding system in figure 3. As for DS_1, the one in figure 4 is a logically reversible system.

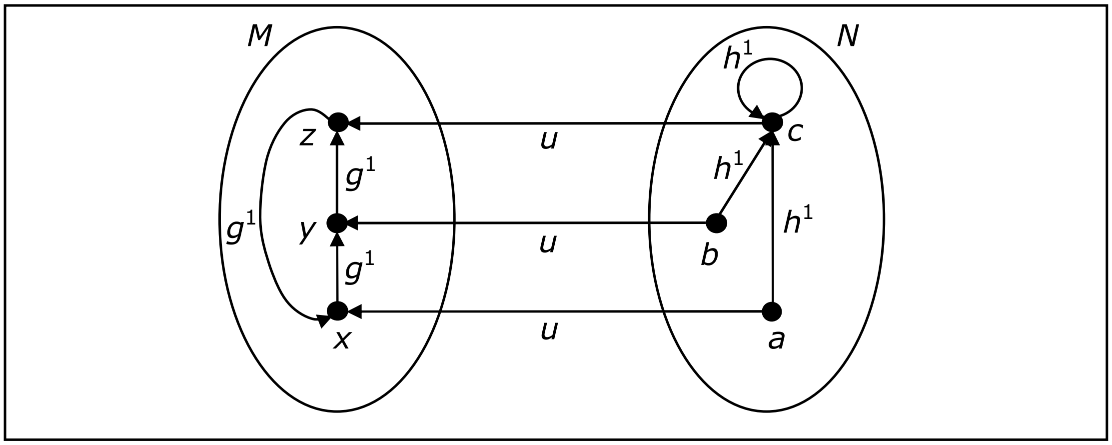

FIGURE 4 DS_1 emulates DS_2, DS_1 is logically reversible (thus, not strongly irreversible), and DS_2 is strongly irreversible

It is perhaps worth pointing out that figures 3 and 4 might somehow mislead the reader, for they do not explicitly show the *whole* dynamical structure of either DS_1 or DS_2, but just the structure of their first state-transitions g^1 and h^1. This may give the wrong impression that the mapping u is blind to some of the dynamical structure of DS_2. In particular, one might argue, the forward fork structure that is present in the dynamics of DS_2 is nowhere to be found in the dynamics of DS_1, so that u is completely blind to dynamical structure of this kind. Therefore, the mapping u is clearly dynamically partial; it follows that it cannot be invoked as a reliable basis for the claim that emergence and reduction can co-occur.

But this kind of argument relies on a false premise, for it is not true that the forward fork structure of DS_2 is nowhere to be found in the dynamics of DS_1. This is clear when the relevant parts of the dynamical structure *implicit* in figure 3 or 4 are explicitly drawn, like in figures 3A and 4A below. Figure 3A differs from figure 3 just for the explicit representation of the second state transition g^2 of DS_1, when it is applied to state x. Figure 4A differs from figure 4 just for the explicit representation of both $g^2(x)$ and $g^3(z)$. A quick look at the two figures shows that DS_1 *indeed contains* a forward fork structure that corresponds, through u, to the forward fork of DS_2, even though the DS_1 fork is embedded in some additional structure, and its labeling is not completely homologous to the labeling of the DS_2 fork.[13]

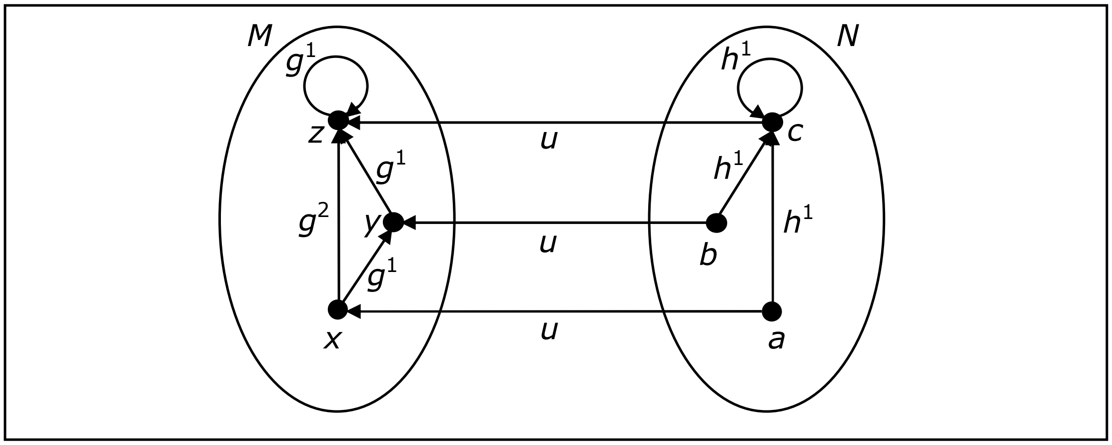

FIGURE 3A DS_1 emulates DS_2, DS_1 is logically irreversible but not strongly irreversible, and DS_2 is strongly irreversible

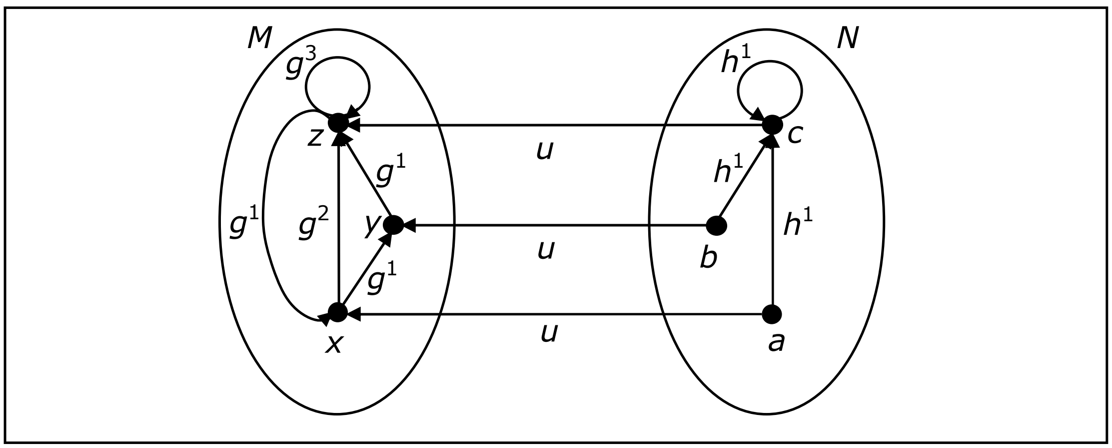

FIGURE 4A DS_1 emulates DS_2, DS_1 is logically reversible (thus, not strongly irreversible), and DS_2 is strongly irreversible

Yet, one might still maintain that this response is only adequate from a purely formal point of view, but it does not carry over to empirically interpreted models. The essence of preserving the forward fork structure of DS_2 in the state-transition relationships of DS_1 is to distinguish $g^2: u(a) \rightarrow u(c)$ from $g^1: u(b) \rightarrow u(c)$. But in fact [*] $g^2: u(a) \rightarrow u(c) = (g^1: u(a) \rightarrow u(b)) \circ (g^1: u(b) \rightarrow u(c))$. If we think of both DS_2 and DS_1 as empirically interpreted models, then surely "=" is to be read "physically identical" and when thus read physically, [*] says that the $u(a) \rightarrow u(c)$ time evolution is only achieved by passing through $u(b)$, precisely what is physically disallowed if the irreversible dynamics of the forward fork structure of DS_2 is to be respected. For this latter $g^2: u(a) \rightarrow u(c)$ would have to represent a dynamically distinct process from $g^1: u(b) \rightarrow u(c)$, and so [*] fails. Thus, even though, in a purely formal sense, the forward fork structure of DS_2 is indeed preserved in DS_1, this is not sufficient to respect the physically relevant aspects of such a structure.

This line of argument is intuitively appealing, but a closer scrutiny shows that (i) it either involves an equivocation fallacy, or (ii) it begs the question. The argument purports to show that the physical reading of [*] entails that $g^2: u(a) \rightarrow u(c)$ and $g^1: u(b) \rightarrow u(c)$ are not physically distinct processes, so that they cannot provide a correct representation of the forward fork structure of DS_2. However, the physical reading of [*] only entails that $g^2: u(a) \rightarrow u(c)$ contains $g^1: u(b) \rightarrow u(c)$ as a proper subprocess and this, in turn, entails that $g^2: u(a) \rightarrow u(c)$ and $g^1: u(b) \rightarrow u(c)$ are not *completely* distinct physical processes, like the ones in the original DS_2 fork $h^1: a \rightarrow c$ and $h^1: b \rightarrow c$. But it surely does not entail that $g^2: u(a) \rightarrow u(c)$ and $g^1: u(b) \rightarrow u(c)$ are not physically *distinct*. In fact, they *are* distinct, for $g^1: u(b) \rightarrow u(c)$ is a *proper* subprocess of $g^2: u(a) \rightarrow u(c)$. On the other hand, while we can surely grant that preserving the forward fork structure of DS_2 requires that $g^2: u(a) \rightarrow u(c)$ and $g^1: u(b) \rightarrow u(c)$ be physically distinct processes, the stronger requirement of *complete* distinction would beg the question. For this assumption would imply the strong irreversibility of DS_1in the first place.[14]

6 Toward a general representational theory of reduction and emergence

Traditionally, reduction has been analyzed in terms of a *deductive* relationship between two empirically interpreted formal *theories*, via correspondence rules, or bridge principles, between the terms of the two theories (Nagel 1961). Schaffner's *General Reduction Paradigm* (1967) was an early attempt to modify Nagel's classic account, so as to accomodate cases where the reduced theory is, strictly speaking, false.[15] The most comprehensive and detailed deductivist account of reduction is Churchland and Hooker's *Imaging Approach* (Churchland 1979, 1985; Hooker 1979, 1981, 2005), which can be seen as a creative development of Nagel's basic insights, as well as a sensible departure from Nagel's explicit tenets (Beckermann 1992; Bickle 1998, 2003; Marras 2002). By shifting attention from formal theories to mathematical *models*, it is natural to think of reduction in terms of some kind of *representation* relationship between two models. This paper has argued so far that, if the two models are dynamical systems (in the purely mathematical sense), the relationship of emulation is sufficient for reduction.

An important point needs to be stressed. If we think of S_2's reduction to S_1 as a form of *deduction* of theory S_2 from theory S_1 (more precisely, the deduction of S_2 from S_1 in conjunction with bridge principles *BP*, where such principles are interpreted as contingent identities between the properties referred to by the primitive terms of S_2 and the properties referred to by the corresponding terms of S_1), then it is obvious that all the properties of S_2 are a fortiori properties of S_1. Therefore, if we take *this kind* of approach to reduction, there cannot be two theories S_2 and S_1 such that S_2 is reduced to S_1 and S_2 has emergent properties with respect to S_1.

The simple form of the foregoing argument cannot be applied to the Imaging Approach. As it is well known, this approach does not view reduction as a plain *Yes* or *No* matter; rather, it envisages a two-dimensional continuum along the two dimensions of ontological retention/elimination and exact/approximate law-mimicry. Also, the deductive relationship between S_1 and S_2 is not direct. For it is not S_2 that is deduced from $S_1 \cup BP$, but an image of S_2, S_2^*, which is expressed in the vocabulary of S_1 (or, more generally, in the vocabulary of a definitional extension of S_1) and is deduced from $S_1 \cup C$, where C is a specific set of reduction conditions, expressed in the vocabulary of S_1; such conditions may very well include *counterfactual* or physically unreal idealizations or limiting assumptions.

The location of a reduction on a specific point of the two-dimensional continuum is then determined, essentially, by an answer to the following two (quite complex) questions: First, how much faithful to S_2 is the image S_2^*? And, second, to what degree are the conditions in C counterfactual? Exact law-mimicry with complete retention will obtain with maximum faithfulness (isomorphism[16] between S_2 and S_2^*) and zero counterfactual degree for C. Exact law-mimicry with complete elimination will obtain with maximum S_2^* faithfulness and maximum counterfactual degree for C. In general, the faithfulness degree of S_2^* will locate a reduction on a specific point of the law-mimicry dimension, but it may also contribute to the position on the ontological retention/elimination spectrum;[17] the counterfactual degree of C will then definitely fix the position on such spectrum.

Thus, according to the Imaging Approach, a successful reduction of S_2 to S_1 yields the *replacement* of S_2 with a sufficiently faithful image S_2^*, whose ontology, in general, only *partially* retains the ontology of S_2. Also note that all those properties or entities of S_2 that are not retained by S_2^* are just eliminated, for they are recognized as non-existent. But then, if S_2 is reduced to S_1, we cannot in general conclude that all the properties of S_2 are also properties of S_1; for this conclusion holds only if the reduction is completely retentive. In general, however, we can still conclude that all the *retained* properties of S_2 are also properties of S_1. Since these are all the properties of S_2 that S_1 recognizes as existent, we can also affirm that, for the Imaging Approach, there cannot be two theories S_2 and S_1 such that S_2 is reduced to S_1 and S_2 has emergent properties with respect to S_1.

Marras (2002) has convincingly argued that Kim's *Functionalizing Approach* to reduction (1998) is in fact a version of Nagel's account; such version is essentially equivalent to the Imaging Approach. Thus, if Marras is right, the conclusion of the preceding paragraph holds for the Functionalizing Approach as well. More generally, we should notice that Kim explicitly criticizes Nagel's classic account for not being able to *identify* the properties of the reduced theory with the corresponding properties of the reducing one (Kim 1998, ch. 4). Thus, according to Kim, such an identification is a requirement that an adequate account of reduction should satisfy. But then, on Kim's view, reduction and emergence cannot simultaneously occur.

The same conclusion holds for Hooker's (2004) *dynamically based* development of the Imaging Approach (henceforth, *DIA*). According to *DIA*,

> The dynamical characterization of reduction not clearly holding is that of degeneracy of relationship in the limit domain, while the characterization of emergence is of the formation of a top-down dynamical constraint—thereby also ensuring degeneracy of relationship. (Hooker 2004, 457-8)

Since the formation of a top-down dynamical constraint also ensures degeneracy of relationship, *DIA* excludes the simultaneous occurrence of emergence and reduction (also see Hooker 2004, 457, table 3).

But this need not be the case if we think of reduction not as a *deductive* relationship between two formal *theories*,[18] but as a form of *representation* between two mathematical *models* MS_1 and MS_2, which grants the retrieval, within the representing model MS_1, of an isomorphic image[19] of MS_2. In fact, as I have just shown for the special case of dynamical systems, this view of reduction is compatible with the existence of structural properties of the reduced system that are not also properties of the reducing one. Therefore, under this view, reduction and emergence are no longer incompatible relationships but, rather, complementary ones.

7 Models of phenomena—conditions for reduction

Thus far, the *representational theory* of reduction has a precise formulation only if the models involved are dynamical systems in the purely mathematical sense. However, we have seen in sec. 3 that dynamical systems can also be intended as *models of real phenomena*. According to this second sense of the term "model", a dynamical system is not a purely mathematical entity DS; rather, it is a pair (DS, I_H), where I_H is an empirical interpretation that links the purely mathematical model DS to a phenomenon H. The representational theory should then be further developed to provide conditions for reduction of an empirically interpreted dynamical system (DS_2, I_{H_2}) to another one (DS_1, I_{H_1}). I will briefly sketch here the main lines of such development. The following exposition has no pretention to exhaustiveness. Its goal is just to trace a possible way along which an adequate representational theory of reduction for *empirically interpreted* dynamical systems might be worked out.

In general, a *phenomenon* H can be characterized by a pair (F, B_F) of two distinct elements. The first one, F, is a *functional description* of (i) an abstract type of real system AS_F and (ii) a general spatio-temporal scheme CS_F of its causal interactions; the second element, B_F, is the set of all concrete systems of type AS_F that also satisfy the causal interaction scheme CS_F; B_F is called the *application domain*[20] of the phenomenon H. In particular, the functional description of the abstract system AS_F specifies its structural elements (or functional parts) and their mutual relationships and organization, while the description of the causal scheme CS_F specifies the initial conditions of AS_F's evolution.

For example, let $H_e = (F_e, B_{F_e})$ be the phenomenon of the free fall of a medium size body in the vicinity of the earth (from now on, I will refer to H_e just as *the phenomenon of free fall*). In this case, the abstract type of real system AS_{F_e} has just one structural element, namely, a medium size body in the vicinity of the earth. The causal interaction scheme CS_{F_e} consists in releasing the body at an arbitrary instant, and with a *vertical* velocity (relative to the earth's surface) and position whose respective values are within appropriate boundaries. B_{F_e} is then the set of all concrete medium size bodies in the vicinity of the earth that satisfy the given scheme of causal interactions.

Let $DS = (X_1 \times \ldots \times X_n, (g^t)_{t \in T})$ be a dynamical system whose state space has n components X_i ($1 \leq i \leq n$, $i \in Z^+$). An *interpretation* I_H *of DS on a phenomenon H* consists in identifying each component X_i with the set of all possible values of a magnitude $\boldsymbol{M}_i$ of the phenomenon H, and the time set T with the set of all possible instants of the time $\boldsymbol{T}$ of H itself. An interpretation I_H of DS on H is *empirical* if the time $\boldsymbol{T}$ and all the magnitudes $\boldsymbol{M}_i$ are measurable properties of the phenomenon H. A pair (DS, I_H), where DS is a dynamical system with n components and I_H is an empirical interpretation of DS on H, is said to be a *model of the phenomenon H*. Such a model is said to be *empirically correct* if, for any i, all measurements of magnitude $\boldsymbol{M}_i$ are consistent with the corresponding values x_i determined by DS.

As an example, let us consider again the phenomenon of free fall H_e. Let DS_e be the dynamical system with two components specified in sec. 3, and I_{H_e} be its interpretation given in sec. 3; then, according to the previous definitions, I_{H_e} is an empirical interpretation of DS_e on H_e, and (DS_e, I_{H_e}) is a model of H_e. For an appropriate value of the constant $\boldsymbol{k}$ (see note 5), such a model also turns out to be empirically correct.

Let us now consider two phenomena $H_1 = (F_1, B_{F_1})$ and $H_2 = (F_2, B_{F_2})$, and two empirically interpreted dynamical systems $\boldsymbol{DS}_1 = (DS_1, I_{H_1})$ and $\boldsymbol{DS}_2 = (DS_2, I_{H_2})$ such that $\boldsymbol{DS}_1$ is a model of H_1 and $\boldsymbol{DS}_2$ is a model of H_2. What are the conditions for reduction of $\boldsymbol{DS}_2$ to $\boldsymbol{DS}_1$? I will divide the discussion into three distinct cases.

CASE 1 Let us suppose that $B_{F_2} \subseteq B_{F_1}$. Under this hypothesis, it seems sensible to claim that, if DS_1 emulates DS_2, then $\boldsymbol{DS}_2$ *is reduced to* $\boldsymbol{DS}_1$. To see this point, let us notice, first, that the hypothesis $B_{F_2} \subseteq B_{F_1}$ ensures that any concrete system described by $\boldsymbol{DS}_2$ is also described by $\boldsymbol{DS}_1$. Second, let u: $Y_1\times\ldots\times Y_m \to X_1\times\ldots\times X_n$ be an emulation of $DS_2 = (Y_1\times\ldots\times Y_m, (h^v)_{v\in V})$ in $DS_1 = (X_1\times\ldots\times X_n, (g^t)_{t\in T})$. Thus, by def. [8], any state transition h^v: $(y_1, \ldots, y_n) \to (y_1', \ldots, y_n')$ corresponds to a state transition g^t: $(x_1, \ldots, x_n) \to (x_1', \ldots, x_n')$, where $u(y_1, \ldots, y_n) = (x_1, \ldots, x_n)$ and $u(y_1', \ldots, y_n') = (x_1', \ldots, x_n')$. In addition, since $\boldsymbol{DS}_2$ is a model of H_2, for any j, y_j and y_j' are values of a measurable magnitude $\boldsymbol{M}_j$ of H_2, and v is an instant of the time $\boldsymbol{T}_2$ of H_2; on the other hand, since $\boldsymbol{DS}_1$ is a model of H_1, for any i, x_i and x_i' are values of a measurable magnitude $\boldsymbol{M}_i$ of H_1, and t is an instant of the time $\boldsymbol{T}_1$ of H_1. For any concrete system $RS \in B_{F_2}$, both the $\boldsymbol{DS}_2$ and the $\boldsymbol{DS}_1$ descriptions apply to RS. But then, the emulation function u tells us exactly how the $\boldsymbol{DS}_2$ description of RS corresponds to the $\boldsymbol{DS}_1$ description.

As an example, let $\boldsymbol{DS}_e = (DS_e, I_{H_e})$, where H_e = the phenomenon of free fall. Let $H_p = (F_p, B_{F_p})$ be *the phenomenon of projectile motion*, where its functional description F_p and its application domain B_{F_p} are specified as follows. The abstract type of real system AS_{F_p} is a medium size body in the vicinity of the earth, and it is thus identical to AS_{F_e}. However, its causal interaction scheme CS_{F_p} is more general than CS_{F_e}, for it consists in the body's being released at an arbitrary instant, and with *any velocity* (relative to the earth) and position whose respective values are within appropriate boundaries. B_{F_p} is then the set of all concrete medium size bodies in the vicinity of the earth that satisfy the given more general scheme of causal interactions.

Let us then consider the following system of four ordinary differential equations $\langle da(t)/dt = 0,\ dc(t)/dt = a(t),\ dx(t)/dt = \boldsymbol{k},\ dy(t)/dt = x(t)\rangle$, where $\boldsymbol{k}$ is a fixed real non-negative constant. The solutions of such equations uniquely determine the dynamical system $DS_p = (A\times C\times X\times Y, (g^t)_{t\in T})$, where $A = C = X = Y = T = R$ (the real numbers) and, for any $t, a, c, x, y \in R$, $g^t(a, c, x, y) = (a, at + c, \boldsymbol{k}t + x, \boldsymbol{k}t^2/2 + xt + y)$. Let I_{H_p} be the following interpretation of DS_p on H_p. The first component A of the state space of DS_p is the set of all possible values of the *horizontal velocity* of an arbitrary projectile, the second component C is the set of all possible values of the *horizontal position* of the projectile, the third component X is the set of all possible values of its *vertical velocity*, the fourth component Y is the set of all possible values of its *vertical position*, and the time set T of DS_p is the set all possible instants of *physical time*. Since all five of these magnitudes are measurable or detectable properties of the intended phenomenon H_p, I_{H_p} is an *empirical* interpretation of DS_p on H_p.

Let $\boldsymbol{DS}_p = (DS_p, I_{H_p})$. By the respective definitions of B_{F_e} and B_{F_p}, $B_{F_e} \subset B_{F_p}$. Thus, by case 1, to show that $\boldsymbol{DS}_e$ is reduced to $\boldsymbol{DS}_p$, it suffice to exhibit an emulation u of DS_e in DS_p. Let u: $X\times Y \to A\times C\times X\times Y$ and, for any $x, y \in R$, $u(x, y) = (0, 0, x, y)$; then, quite obviously, u is an emulation of DS_e in DS_p.

CASE 2 Let us suppose next that $B_{F_2} \cap B_{F_1} = \varnothing$. In this case, no matter how DS_1 and DS_2 are related, $\boldsymbol{DS}_2$ *is not reduced to* $\boldsymbol{DS}_1$. For, even if DS_2 is identical to DS_1, *any* concrete system described by $\boldsymbol{DS}_2$ (that is to say, any concrete system $RS \in B_{F_2}$) is not a system also described by $\boldsymbol{DS}_1$.

CASE 3 The case $B_{F_2} \cap B_{F_1} \neq \varnothing$ and $\neg(B_{F_2} \subseteq B_{F_1})$ is still left. This case is a combination of the previous two. In fact, for some concrete system $RS \in B_{F_2}$, both the $\boldsymbol{DS}_2$ and the $\boldsymbol{DS}_1$ descriptions apply to RS; however, if $RS \in B_{F_2}$ and $RS \notin B_{F_1}$, only the $\boldsymbol{DS}_2$ description applies to RS. Thus, in this case, if DS_1 emulates DS_2, $\boldsymbol{DS}_2$ *is incompletely reduced to* $\boldsymbol{DS}_1$.

We have just seen that case 3 only grants *incomplete* reduction of $\boldsymbol{DS}_2$ to $\boldsymbol{DS}_1$, provided that DS_1 emulates DS_2. However, $\boldsymbol{DS}_2$ may turn out to be *multiply* reduced to a *family* $(\boldsymbol{DS}_j)_{j\in J} = ((DS_j, I_{H_j}))_{j\in J}$ of empirically interpreted dynamical systems, each of which satisfies case 3 and emulates DS_2. This will be the case if the application domain B_{F_2} is included in the union of all application domains B_{F_j}. More precisely, for $\boldsymbol{DS}_2$ to *be multiply reduced to* $(\boldsymbol{DS}_j)_{j\in J}$, it is sufficient that, for any $j \in J$, $B_{F_2} \cap B_{F_j} \neq \varnothing$, $\neg(B_{F_2} \subseteq B_{F_j})$, DS_j emulates DS_2, and $B_{F_2} \subseteq \bigcup_{j\in J} B_{F_j}$.

A relationship between this condition for multiple reduction and the second order property version of the multiple realization concept (Kim 1998, 19-20, 103-4) is worth noticing. According to the latter, a property P *is multiply realized by properties of type D* just in case, for any x, x has P iff there is a property P_j of type D such that x has P_j. Any property P_j that satisfies the previous condition is said a *D-realizer* of the property P, and the property P itself is said a *second order property*.

Suppose now that $\boldsymbol{DS}_2$ is multiply reduced to $(\boldsymbol{DS}_j)_{j\in J}$ according to the previously stated sufficient condition. Let P_2 be the property that corresponds to functional description F_2 and, for any $j \in J$, P_j be the property that corresponds to functional description F_j. Let D be the property of being one of the properties P_j, for some $j \in J$. As $B_{F_2} \subseteq \bigcup_{j\in J} B_{F_j}$, it follows that, if x has P_2, then x has P_j, for some $j \in J$. Furthermore, if $B_{F_2} = \bigcup_{j\in J} B_{F_j}$, the converse holds as well, so that P_2 is multiply realized by properties of type D, and $(P_j)_{j\in J}$ is the family of its D-realizers.

From an intuitive point of view, the emulation relationship holds between two dynamical systems DS_1 and DS_2 when the *whole* dynamics of DS_2 is *exactly* reproduced by DS_1. I have argued so far that this relationship might be the basis for a new approach to reduction, which I have called *representational*. However, it is well known that, in many cases of inter-theoretic reduction, the relationship between the reduced theory S_2 and the reducing one S_1 is such that S_2 is only *partially* and *approximately* reduced to S_1. Furthermore, such a relationship typically is an *asymptotic* one, that is, it depends on some parameter $\boldsymbol{p}_1$ of S_1 in such a way that, for $\boldsymbol{p}_1$ tending to some fixed limiting value $\boldsymbol{p}$, S_2 tends to be partially and approximately reduced to S_1, as established according to the limiting value $\boldsymbol{p}$.[21]

The simple form of the emulation relationship considered so far may very well be the basis for a representational account of *total* and *exact* reduction (like, for example, the reduction of $\boldsymbol{DS}_e$ to $\boldsymbol{DS}_p$; see case 1 above). Nevertheless, we need a more sophisticated version of emulation for dealing with cases of asymptotic, partial and approximate reduction. In the Appendix, I suggest how this might be accomplished and provide (i) a formal definition of *partial* and *approximate* emulation and (ii) a simple example that shows how this relationship may turn out to be asymptotic.

8 Concluding remarks

I have argued in this paper that reduction is better analyzed in terms of a *representational* relationship between *models*, rather than a *deductive* relationship between *theories*. Contrary to the received view, reduction and emergence may co-occur, and in fact they can be thought as complementary manifestations of an underlying representational relationship between mathematical models, namely, the one of *emulation*.

The representational theory of reduction and emergence has been developed so far only for the special case of dynamical systems (either empirically interpreted, or not). However, even in this special form, the theory is far from being complete. I will mention here just two basic points that should be further investigated and expanded.

First, as far as empirically interpreted dynamical systems are concerned, only conditions for reduction, but no conditions for constitution, have been proposed. Therefore, until constitution conditions are given, any claim concerning emergence is, strictly speaking, limited to dynamical systems as purely mathematical entities.

Second, the emulation relationship between two dynamical systems $DS_1 = (M, (g^t)_{t\in T})$ and $DS_2 = (N, (h^v)_{v\in V})$ has been considered in two different forms (def. [8] and [17]), which are respectively based on a *total* and *exact* structure preserving mapping u, or on a *partial* and *approximate* one. The crucial point is that the mapping u preserve (exactly or approximately) DS_2's structure (the whole or a part) in DS_1's structure, and this has been obtained by taking u to be an injective function from N to M. Yet, it is possible to obtain the same result by either dropping the injectivity requirement, or by taking u to be a function from M to N. Therefore, the whole theory developed so far should be revised and completed in the light of these more general assumptions on the structure preserving mapping u.

Nevertheless, even a complete representational theory for dynamical systems would not be sufficient to account for all relevant cases of reduction or emergence, for many models in real science are not of this kind. What we need is a *general* representational theory, as precise as the one restricted to dynamical systems, which apply to *arbitrary models*. The formulation of such a general theory, however, is not an easy matter, for it involves a preliminary investigation of fairly hard questions like: What is, *in general*, a purely mathematical model?[22] What is a structure preserving mapping between two *arbitrary* mathematical models? What is the relationship between two *arbitrary* mathematical models that generalizes the one of emulation between dynamical systems? What is, *in general*, an empirical interpretation of a mathematical model on a phenomenon?

Appendix Partial and approximate emulation—weaker conditions for reduction

Intuitively, a dynamical system $DS_1 = (M, (g^t)_{t\in T})$ partially emulates a second dynamical system $DS_2 = (N, (h^v)_{v\in V})$ if DS_1 exactly reproduces the dynamics of DS_2, limited to a fixed subset C of DS_2's state space N. This concept is thus a straightforward relativization of def. [8]. Let $C \subseteq N$, and define:

[15] DS_1 *C-emulates* DS_2 iff there is an injective function u: $N \to M$ such that for any $v \in V$, for any $c \in C$, there is $t \in T$ such that $u(h^v(c)) = g^t(u(c))$. Any function u that satisfies the previous condition is called a *C-emulation of* DS_2 *in* DS_1.

Intuitively, DS_1 approximately emulates DS_2 if each state transition h^v: $y \to y$'of DS_2 approximately corresponds to a state transition g^t: $x \to x$' of DS_1. This idea can be made precise by requiring that, for some injective function u, $u(y) = x$, and $u(y')$ be *sufficiently close* to x', where the two states $u(y')$, $x' \in M$ are sufficiently close to each other if their *distance* does not exceed a fixed non-negative real δ. Thus, the concept of approximate emulation in fact presupposes that M (i.e., the state space of DS_1) be equipped with a metric. Let d: $M \times M \to R^+$ be a metric on M, let $\delta \in R^+$. We then define:

[16] DS_1 *δ-emulates* DS_2 iff there is an injective function u: $N \to M$ such that, for any $v \in V$, for any $c \in N$, there is $t \in T$ such that $d(u(h^v(c)), g^t(u(c))) \le \delta$. Any function u that satisfies the previous condition is called a *δ-emulation of* DS_2 *in* DS_1.

Finally, by combining definitions [15] and [16], we get a definition of the intuitive idea of *partial* and *approximate* emulation. Let $C \subseteq N$, d: $M \times M \to R^+$ be a metric on M, and $\delta \in R^+$;

[17] DS_1 *C-δ-emulates* DS_2 iff there is an injective function u: $N \to M$ such that for any $v \in V$, for any $c \in C$, there is $t \in T$ such that $d(u(h^v(c)), g^t(u(c))) \le \delta$. Any function u that satisfies the previous condition is called a *C-δ-emulation of* DS_2 *in* DS_1.

1 *An example*

Let $X = Y = R$ (the real numbers), $T = V = R^+$ (the non-negative real numbers), and $DS_{n_{\boldsymbol{l}}} = (X \times Y, (g^t_{\boldsymbol{l}})_{t \in T})$ be the dynamical system that is determined by the solutions of the following non-linear system of ordinary differential equations $\langle dx(t)/dt = y(t),\ dy(t)/dt = -\boldsymbol{k}\, sin(x(t)/\boldsymbol{l})\rangle$, where $\boldsymbol{k}$ is a fixed real positive constant, and $\boldsymbol{l}$ is an arbitrary real positive parameter; note that this system is in fact a *non-linear classic pendulum*. On the other hand, let $DS_{o_{\boldsymbol{l_0}}} = (X \times Y, (h^v_{\boldsymbol{l_0}})_{v \in V})$ be the dynamical system that is determined by the solutions of the following linear system of ordinary differential equations $\langle dx(v)/dv = y(v),\ dy(v)/dv = -\boldsymbol{k}x(v)/\boldsymbol{l_0}\rangle$, where $\boldsymbol{k}$ is as above, and $\boldsymbol{l_0}$ is an arbitrary real positive parameter; this second system is a *harmonic oscillator*.[23]

Let $0 \le \theta \le \pi$, and $C_{\boldsymbol{l_0},\theta} = \{c$ such that, for some $x, y \in R$, $c = (x, y)$, $y = 0$ and $-\theta \le x/\boldsymbol{l_0} \le \theta\}$. As it can be visually verified by means of any dynamical systems software, for an appropriately chosen $\delta_{\boldsymbol{l_0},\theta} > 0$, for any $v \in V$, for any $c \in C_{\boldsymbol{l_0},\theta}$, $d(h^v_{\boldsymbol{l_0}}(c), g^v_{\boldsymbol{l_0}}(c)) \le \delta_{\boldsymbol{l_0},\theta}$, where d is the usual Euclidean distance on $X \times Y = R^2$. Let $\delta_{\boldsymbol{l_0},\theta^{min}}$ be the minimum of all $\delta_{\boldsymbol{l_0},\theta}$ such that, for any $v \in V$, for any $c \in C_{\boldsymbol{l_0},\theta}$, $d(h^v_{\boldsymbol{l_0}}(c), g^v_{\boldsymbol{l_0}}(c)) \le \delta_{\boldsymbol{l_0},\theta}$. Note that $\delta_{\boldsymbol{l_0},\theta^{min}}$ must exist because R satisfies the least upper bound property.[24] Let u be the identity function on $X\times Y$. By def. [17], it thus follows that u is a $C_{\boldsymbol{l}_0,\theta}$-$\delta_{\boldsymbol{l}_0,\theta^{min}}$-emulation of $DS_{o_{\boldsymbol{l}_0}}$ in $DS_{n_{\boldsymbol{l}_0}}$, and so $DS_{n_{\boldsymbol{l}_0}}$ $C_{\boldsymbol{l}_0,\theta}$-$\delta_{\boldsymbol{l}_0,\theta^{min}}$-emulates $DS_{o_{\boldsymbol{l}_0}}$.

It is important to keep in mind that $\delta_{\boldsymbol{l}_0,\theta^{min}}$ represents the approximation degree to which $DS_{n_{\boldsymbol{l}_0}}$ partially emulates $DS_{o_{\boldsymbol{l}_0}}$ with respect to domain $C_{\boldsymbol{l}_0,\theta}$. Besides, $\delta_{\boldsymbol{l}_0,\theta^{min}}$ is a function of $\theta \in [0, \pi]$. Therefore, we can study the behavior of $\delta_{\boldsymbol{l}_0,\theta^{min}}$ for θ tending to 0 from the right, and it is not difficult to verify that $\lim_{\theta\to 0^+} \delta_{\boldsymbol{l}_0,\theta^{min}} = 0$. That is to say, for θ tending to 0 from the right, the approximation degree to which $DS_{n_{\boldsymbol{l}_0}}$ partially emulates $DS_{o_{\boldsymbol{l}_0}}$ with respect to domain $C_{\boldsymbol{l}_0,\theta}$ tends to 0.

We can also verify that, for any $\boldsymbol{l}$, and for an appropriately chosen $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta} > 0$, it holds that, for any $v \in V$, for any $c \in C_{\boldsymbol{l}_0,\theta}$, $d(h^v_{\boldsymbol{l}_0}(c), g^v_{\boldsymbol{l}}(c)) \leq \delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta}$. Let $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ be the minimum of all $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta}$ such that, for any $v \in V$, for any $c \in C_{\boldsymbol{l}_0,\theta}$, $d(h^v_{\boldsymbol{l}_0}(c), g^v_{\boldsymbol{l}}(c)) \leq \delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta}$. Note that $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ must exist because R satisfies the least upper bound property. Let u be the identity function on $X\times Y$. By def. [17], it thus follows that u is a $C_{\boldsymbol{l}_0,\theta}$-$\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$-emulation of $DS_{o_{\boldsymbol{l}_0}}$ in $DS_{n_{\boldsymbol{l}}}$, and so $DS_{n_{\boldsymbol{l}}}$ $C_{\boldsymbol{l}_0,\theta}$-$\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$-emulates $DS_{o_{\boldsymbol{l}_0}}$.

Let us notice now that $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ represents the approximation degree to which $DS_{n_{\boldsymbol{l}}}$ partially emulates $DS_{o_{\boldsymbol{l}_0}}$ (with respect to domain $C_{\boldsymbol{l}_0,\theta}$); also, $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ is a function of $\boldsymbol{l} \in R^+$-{0}, and it thus makes sense to study the asymptotic behavior of $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ for $\boldsymbol{l} \to \boldsymbol{l}_0$. In fact, if $0 \leq \theta < \pi$, it can be verified that $\lim_{\boldsymbol{l}\to\boldsymbol{l}_0} \delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}} = \delta_{\boldsymbol{l}_0,\theta^{min}}$. And this means that, for $\boldsymbol{l}$ tending to $\boldsymbol{l}_0$, the approximation degree to which $DS_{n_{\boldsymbol{l}}}$ partially emulates $DS_{o_{\boldsymbol{l}_0}}$ (with respect to domain $C_{\boldsymbol{l}_0,\theta}$) tends to the approximation degree to which $DS_{n_{\boldsymbol{l}_0}}$ partially emulates $DS_{o_{\boldsymbol{l}_0}}$ (with respect to the same domain $C_{\boldsymbol{l}_0,\theta}$). In this precise sense, then, the relationship of partial and approximate emulation of $DS_{o_{\boldsymbol{l}_0}}$ by $DS_{n_{\boldsymbol{l}}}$ (with respect to domain $C_{\boldsymbol{l}_0,\theta}$ and to approximation degree $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$) turns out to be asymptotic.

It is also interesting to understand exactly how $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ approaches its limiting value $\delta_{\boldsymbol{l}_0,\theta^{min}}$ when $\boldsymbol{l} \to \boldsymbol{l}_0$, for the asymptotic behavior of $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ is not uniform, but it depends on specific regions of $\boldsymbol{l}$'s domain R^+-{0}. If $\boldsymbol{l}$ approaches $\boldsymbol{l}_0$ from the right in the interval $[\boldsymbol{l}_0, +\infty]$, then $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ tends to the limiting value $\delta_{\boldsymbol{l}_0,\theta^{min}}$, and an analogous behavior obtains for $\boldsymbol{l}$ tending to $\boldsymbol{l}_0$ from the left in the interval $[\theta\boldsymbol{l}_0/\pi, \boldsymbol{l}_0]$. For $\boldsymbol{l}$ approaching $\theta\boldsymbol{l}_0/m\pi$ from the left in any interval $[\theta\boldsymbol{l}_0/(m+2)\pi, \theta\boldsymbol{l}_0/m\pi]$ (where $m \in Z^+$ and m is odd), $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ tends to a limiting value $\delta^-_{m\pi}$. However, for $\boldsymbol{l}$ approaching $\theta\boldsymbol{l}_0/(m+2)\pi$ from the right in any interval $[\theta\boldsymbol{l}_0/(m+2)\pi, \theta\boldsymbol{l}_0/m\pi]$ (where $m \in Z^+$ and m is odd), $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ tends to a limiting value $\delta^+_{(m+2)\pi}$, where $\delta^+_{(m+2)\pi} < \delta^-_{(m+2)\pi}$. Also, for $\boldsymbol{l}$ tending to $\theta\boldsymbol{l}_0/\pi$ from the right in the interval $[\theta\boldsymbol{l}_0/\pi, \boldsymbol{l}_0]$, $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ tends to a limiting value δ^+_π, where $\delta^+_\pi < \delta^-_\pi$. It thus follows that, for any odd $m \in Z^+$, $\delta_{\boldsymbol{l},\boldsymbol{l}_0,\theta^{min}}$ is discontinuous at $\theta\boldsymbol{l}_0/m\pi$.

2 *Empirical interpretations of the two dynamical systems of the previous example*

Both dynamical systems $DS_{n_{\boldsymbol{l}}} = (X \times Y, (g^t_{\boldsymbol{l}})_{t \in T})$ and $DS_{o_{\boldsymbol{l}_0}} = (X \times Y, (h^v_{\boldsymbol{l}_0})_{v \in V})$ can be given natural empirical interpretations on corresponding phenomena. As regards the first system, let $H_{n_{\boldsymbol{l}}} = (F_{n_{\boldsymbol{l}}}, B_{F_{n_{\boldsymbol{l}}}})$ be the phenomenon of the unrestricted swing of a pendulum, where its functional description $F_{n_{\boldsymbol{l}}}$ and its application domain $B_{F_{n_{\boldsymbol{l}}}}$ are specified as follows (from now on, I will refer to $H_{n_{\boldsymbol{l}}}$ as *the phenomenon of pendulum motion*). The abstract type of real system $AS_{F_{n_{\boldsymbol{l}}}}$ (called *simple* or *classic pendulum*) is made up of two structural elements, namely, a light rigid arm of length $\boldsymbol{l}$, with a much heavier "bob" on one of its ends; the arm is pivoted on the other end, so that the bob can frictionlessly swing along a circular path of radius $\boldsymbol{l}$ in a vertical plane. The causal interaction scheme $CS_{F_{n_{\boldsymbol{l}}}}$ consists in releasing the bob at an arbitrary instant and position on its swinging path, with an arbitrary tangent velocity. $B_{F_{n_{\boldsymbol{l}}}}$ is then the set of all concrete simple pendula that satisfy the given scheme of causal interaction. Any such device will be called an *unrestricted pendulum*.

Let $I_{H_{n_{\boldsymbol{l}}}}$ be the following interpretation of $DS_{n_{\boldsymbol{l}}}$ on $H_{n_{\boldsymbol{l}}}$. The first component X of the state space of $DS_{n_{\boldsymbol{l}}}$ is the set of all possible values of the *bob position*[25] of an arbitrary unrestricted pendulum, the second component Y is the set of all possible values of the *bob tangent velocity*, and the time set T of $DS_{n_{\boldsymbol{l}}}$ is the set of all possible instants of *physical time*. Since all three of these magnitudes are measurable or detectable properties of the intended phenomenon $H_{n_{\boldsymbol{l}}}$, $I_{H_{n_{\boldsymbol{l}}}}$ is an empirical interpretation of $DS_{n_{\boldsymbol{l}}}$ on $H_{n_{\boldsymbol{l}}}$, and $\boldsymbol{DS}_{n_{\boldsymbol{l}}} = (DS_{n_{\boldsymbol{l}}}, I_{H_{n_{\boldsymbol{l}}}})$ is thus a model of $H_{n_{\boldsymbol{l}}}$. This model is empirically correct, for an appropriate value of the constant $\boldsymbol{k}$. Henceforth, I will refer to $\boldsymbol{DS}_{n_{\boldsymbol{l}}}$ as an (*unrestricted*) *pendulum model*.

Let $H_{o_{\boldsymbol{l}_0,\theta}} = (F_{o_{\boldsymbol{l}_0,\theta}}, B_{F_{o_{\boldsymbol{l}_0,\theta}}})$ be the phenomenon of pendulum motion restricted to small swings (or, more briefly, *the phenomenon of small pendulum swings*), where its functional description $F_{o_{\boldsymbol{l}_0,\theta}}$ and its application domain $B_{F_{o_{\boldsymbol{l}_0,\theta}}}$ are specified as follows. The abstract type of real system $AS_{F_{o_{\boldsymbol{l}_0,\theta}}}$ is a simple pendulum of length $\boldsymbol{l}_0$, so it is identical to $AS_{F_{n_{\boldsymbol{l}}}}$ (when $\boldsymbol{l} = \boldsymbol{l}_0$). However, the causal interaction scheme $CS_{F_{o_{\boldsymbol{l}_0,\theta}}}$ is more specific than $CS_{F_{n_{\boldsymbol{l}}}}$, for it consists in releasing the pendulum's bob at an arbitrary instant, with zero tangent velocity, and in a position sufficiently close to the intersection O between the swinging path and the vertical straight line r passing through the pendulum pivot. This last clause can be put in the following form. Let θ ($0 \le \theta \le \pi$) be the measure, in radians, of the angle between r and a straight line s passing through the pivot; let x be the bob's releasing position on the swinging path (where the origin is O, and the positive path direction is anticlockwise); then, $-\theta \le x/\boldsymbol{l}_0 \le \theta$. Thus, for θ sufficiently close to 0, the pendulum only performs small swings, when its bob is released at an arbitrary instant, with zero tangent velocity and in position x. $B_{F_{o_{\boldsymbol{l}_0,\theta}}}$ is then the set of all concrete simple pendula that satisfy the given more specific scheme of causal interaction. Any such device will be called a *small swing pendulum*.

Let $I_{H_{o_{\boldsymbol{l}_0,\theta}}}$ be the following interpretation of $DS_{o_{\boldsymbol{l}_0}}$ on $H_{o_{\boldsymbol{l}_0,\theta}}$. The first component X of the state space of $DS_{o_{\boldsymbol{l}_0}}$ is the set of all possible values of the *bob position* of an arbitrary small swing pendulum, the second component Y is the set of all possible values of the *bob tangent velocity*, and the time set V of $DS_{o_{\boldsymbol{l}_0}}$ is the set all possible instants of *physical time*. These three magnitudes are measurable properties of the intended phenomenon $H_{o_{\boldsymbol{l}_0,\theta}}$. Therefore, $I_{H_{o_{\boldsymbol{l}_0,\theta}}}$ is an empirical interpretation of $DS_{o_{\boldsymbol{l}_0}}$ on $H_{o_{\boldsymbol{l}_0,\theta}}$, and $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta}} = (DS_{o_{\boldsymbol{l}_0}}, I_{H_{o_{\boldsymbol{l}_0,\theta}}})$ is a model of $H_{o_{\boldsymbol{l}_0,\theta}}$. Furthermore, if θ is sufficiently small, such a model turns out to be empirically correct (for an appropriate value of the constant $\boldsymbol{k}$). In what follows, $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta}}$ will be called a *small swing pendulum model*.

3 *Weaker conditions for reduction*

Let us now consider a small swing pendulum model $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta}}$ and the corresponding unrestricted pendulum model $\boldsymbol{DS}_{n_{\boldsymbol{l}_0}}$. These two models satisfy case 1 above (sec. 7), for $B_{F_{o_{\boldsymbol{l}_0,\theta}}} \subset B_{F_{n_{\boldsymbol{l}_0}}}$ (by the definitions of the respective application domains $B_{F_{o_{\boldsymbol{l}_0,\theta}}}$ and $B_{F_{n_{\boldsymbol{l}_0}}}$). Moreover, we have previously seen that $DS_{n_{\boldsymbol{l}_0}}$ $C_{\boldsymbol{l}_0,\theta}$-$\delta_{\boldsymbol{l}_0,\theta min}$-emulates $DS_{o_{\boldsymbol{l}_0}}$. The question then naturally arises whether this condition is sufficient for reduction of $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta}}$ to $\boldsymbol{DS}_{n_{\boldsymbol{l}_0}}$.

Let us notice first that $C_{\boldsymbol{l}_0,\theta} \subset X \times Y$ is, on the one hand, the domain with respect to which dynamical system $DS_{n_{\boldsymbol{l}_0}}$ partially emulates $DS_{o_{\boldsymbol{l}_0}}$ and, on the other hand, $C_{\boldsymbol{l}_0,\theta}$ is the set of states of dynamical system $DS_{o_{\boldsymbol{l}_0}}$ that is determined by the specific causal interaction scheme $CS_{F_{o_{\boldsymbol{l}_0,\theta}}}$ of the phenomenon of small pendulum swings. As a consequence, $C_{\boldsymbol{l}_0,\theta}$ can be thought as singling out that part of the structure of $DS_{o_{\boldsymbol{l}_0}}$ that has an empirical interpretation according to $I_{H_{o_{\boldsymbol{l}_0,\theta}}}$. Let us call $E_{C_{\boldsymbol{l}_0,\theta}} = \{e: e = (c, h^{v}_{\boldsymbol{l}_0}(c))$, for some $c \in C_{\boldsymbol{l}_0,\theta}$ and some $v \in V\}$ the *empirical substructure of* $DS_{o_{\boldsymbol{l}_0}}$ *relative to interpretation* $I_{H_{o_{\boldsymbol{l}_0,\theta}}}$.[26] Thus, by def. [17], the *whole* empirical substructure $E_{C_{\boldsymbol{l}_0,\theta}}$ is represented, through a partial emulation function u,[27] by corresponding structure of $DS_{n_{\boldsymbol{l}_0}}$, within approximation degree $\delta_{\boldsymbol{l}_0,\theta min}$. Suppose now that Δ is the best accuracy degree of the measurements relative to interpretation $I_{H_{o_{\boldsymbol{l}_0,\theta}}}$.[28] Then, if $\delta_{\boldsymbol{l}_0,\theta min} \leq \Delta$, we can safely conclude that $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta}}$ is reduced to $\boldsymbol{DS}_{n_{\boldsymbol{l}_0}}$.

In this connection, also recall that $\lim_{\theta \to 0^+} \delta_{\boldsymbol{l}_0,\theta min} = 0$, where $\theta \in [0, \pi]$. This means that the approximation degree $\delta_{\boldsymbol{l}_0,\theta min}$ to which the empirical substructure $E_{C_{\boldsymbol{l}_0,\theta}}$ of $DS_{o_{\boldsymbol{l}_0}}$ is represented by corresponding structure of $DS_{n_{\boldsymbol{l}_0}}$ can be made as small as we please, by taking a sufficiently small value of the angle θ. But then, for a sufficiently small θ_Δ, $\delta_{\boldsymbol{l}_0,\theta_\Delta min} \leq \Delta$, and so $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta_\Delta}}$ is reduced to $\boldsymbol{DS}_{n_{\boldsymbol{l}_0}}$.[29]

In the general case, let $H_1 = (F_1, B_{F_1})$ and $H_2 = (F_2, B_{F_2})$ be two phenomena, and $\boldsymbol{DS}_1 = (DS_1, I_{H_1})$ and $\boldsymbol{DS}_2 = (DS_2, I_{H_2})$ be two empirically interpreted dynamical systems such that $\boldsymbol{DS}_1$ is a model of H_1 and $\boldsymbol{DS}_2$ is a model of H_2. Let us assume that the causal interaction scheme CS_{F_2} of phenomenon H_2 uniquely determines a set of states C_{F_2} of dynamical system DS_2.[30] Let Δ_1 and Δ_2 be, respectively, the best accuracy degrees of the measurements relative to interpretations I_{H_1} and I_{H_2} (see note 28). The previous example thus suggests that case 1 (sec. 7) be supplemented with a weaker sufficient condition for reduction, as follows.

CASE 1A Let us suppose that $B_{F_2} \subseteq B_{F_1}$. If DS_1 C_{F_2}-δ-emulates DS_2, $\delta \leq \Delta_2$ and $\Delta_1 \leq \Delta_2$, then $\boldsymbol{DS}_2$ is reduced to $\boldsymbol{DS}_1$.

A corresponding weaker condition can also be given for the case of *incomplete* reduction (case 3, sec. 7), as follows.

CASE 3A Suppose that $B_{F_2} \cap B_{F_1} \neq \varnothing$ and $\neg(B_{F_2} \subseteq B_{F_1})$. If DS_1 C_{F_2}-δ-emulates DS_2,[31] $\delta \leq \Delta_2$ and $\Delta_1 \leq \Delta_2$, then $\boldsymbol{DS}_2$ *is incompletely reduced to* $\boldsymbol{DS}_1$.

Finally, as regards multiple reduction to a family $(\boldsymbol{DS}_j)_{j\in J} = ((DS_j, I_{H_j}))_{j\in J}$ of empirically interpreted dynamical systems, we get the following weaker condition. For $\boldsymbol{DS}_2$ to *be multiply reduced to* $(\boldsymbol{DS}_j)_{j\in J}$, it is sufficient that, for any $j \in J$, $B_{F_2} \cap B_{F_j} \neq \varnothing$, $\neg(B_{F_2} \subseteq B_{F_j})$, DS_j C_{F_2}-δ_j-emulates DS_2, $\delta_j \leq \Delta_2$, $\Delta_j \leq \Delta_2$, and $B_{F_2} = \bigcup_{j\in J} B_{F_j}$.

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1 The term “discrete dynamical system” is often used (see, for example, Kulenovic and Merino 2002; Martelli 1999; Sandefour 1990) as a synonym for “dynamical system with discrete time”, i.e., according to Szlensk 1984, a *cascade* (see def. [9]). My use of the term “discrete dynamical system” is in accordance with Turing 1950.

2 P is a *structural property of a dynamical system* (or a *dynamical property*) iff for any two mathematical models MS_1 and MS_2, (i) if MS_1 has P, MS_1 is a dynamical system and (ii) if MS_1 has P, and MS_1 is isomorphic to MS_2, then MS_2 has P. Thus, a dynamical property is a property *specific* to dynamical systems that is *preserved* by isomorphism. The proof that any two isomorphic dynamical systems have exactly the same dynamical properties is immediate. Conversely, for any two non-isomorphic dynamical systems DS_1 and DS_2, there is a dynamical property they do not share; namely, the property of being isomorphic to DS_1.

3 By *general dynamical systems theory* I mean the mathematical theory whose Suppes’ style axiomatization (1957, ch. 12) is given by def. [1].

4 I recall that emulation, as defined here, is an exact relationship between two mathematical models; this sense of the term “emulation” is standard in both dynamical systems theory and computation theory, and it should not be confused with a common use of the same term, which refers to the relationship involved in the simulation of a physical system (e.g. a water flow) by a second one (e.g. a digital computer that, by means of appropriate software, implements a mathematical model of the water flow).

5 For an appropriate $\boldsymbol{k}$ (the value of the acceleration due to gravity), the model (DS_e, I_{H_e}) also turns out to be empirically correct (within limits of precision sufficient for many practical purposes). Giunti calls an empirically interpreted dynamical system which is also empirically correct a *Galilean model* (Giunti 1995; Giunti 1997, ch. 3).

6 The condition in the text holds iff any relation σ_i has arity > 0. The general condition is as follows. Let $X = \{x$: for some $i \in I$, $x = \sigma_i$ and σ_i is a relation of arity 0$\}$; then, the whole structure of $(D, (\sigma_i)_{i\in I})$ is the union of X and all relations σ_i of arity > 0. Obviously, this condition reduces to the one in the text when X is empty, i.e., when any relation σ_i has arity > 0.

7 Thus, **C2** is not an *independent* premise of the argument, for it is entailed by def. [1], the standard definition of a mathematical model, and **B2**.

8 The implication from left to right is immediate. The one from right to left is also obvious, once we notice that for any system S_2, *there is* a system S_1 such that $S_1 \neq S_2$ and S_2 is made up of S_1. In fact, take S_1 to be any system with the same constitutive entities as S_2, but in relations different from the ones typical of S_2.

9 With the default choice of the reference system S_1, two standard examples of emergence are the following. (1) The properties of the flight of a flock of birds, as opposed to the properties of the flight of each isolated bird; (2) the properties of the overall evolution of a cellular automaton, as opposed to the properties of the evolution of each constituent finite automaton, taken in isolation.

10 Strong irreversibility is defined in the next section. It is easy to verify that strong irreversibility is a structural property (see note 2) of dynamical systems.

11 When time is discrete (either Z or Z^+, but let us just consider the simpler case of Z^+), dynamical systems reduce to iterated mappings. In fact, on the one hand, if $\boldsymbol{g}: M \rightarrow M$ is an arbitrary function, we can define the n-th ($n > 0$) iteration g^n of g as $g \circ g \circ ... \circ g$ (n times), where $\circ$ is function composition; futhermore, by definition, the 0-th iteration is the identity function. Now, if we take the family $(g^n)_{n\in Z^+}$ of all the n-th iterations of $\boldsymbol{g}$, it is immediate to verify that $(M, (g^n)_{n\in Z^+})$ is a dynamical system in the sense of definition 1 (also note that $g^1 = g$). Conversely, if $(M, (g^n)_{n\in Z^+})$ is a dynamical system whose time set is Z^+, then, by condition 4 of def. 1, and by the definition of family of the n-th iterations, $(g^n)_{n\in Z^+}$ is the family of the n-th iterations of g^1.

[12] Irreversibility is a complex notion, and general dynamical systems theory allows us to make fine distinctions. The use of just a non-negative time-set (either Z^+ or R^+) gives us the weakest and most general concept of an irreversible system.

To understand this, we must take into account the algebraic model $(\{g^t: t \in Z^+\}, \circ)$, i.e., the set of all state transitions together with the composition operation $\circ$. By condition 4 of def. 1, it is immediate to verify that: (1) if $T = Z$ or R, $(\{g^t: t \in Z^+\}, \circ)$ is a commutative group, whose unit is g^0. Furthermore, for any t, the algebraic inverse of g^t (which exists and is unique because $(\{g^t: t \in Z^+\}, \circ)$ is a group) is g^{-t}; this also entails that all state transitions are bijections, and that g^{-t} is the inverse function of g^t (i.e. the two concepts of *algebraic inverse* and *inverse function* coincide); (2) if $T = Z^+$ or R^+, $(\{g^t: t \in Z^+\}, \circ)$ is a commutative monoid, i.e., a commutative semigroup with unit; in this case too the unit is g^0 but, since negative times are lacking, no state transition has an inverse.

This is the situation from the formal point of view. Intuitively, this means that, if we consider just a non-negative time set, the system does not have the internal resources (i.e. the negative state transitions) to retrieve its past from the current state, even though this might be logically possible (it is possible if all state transitions are injective, i.e. if the system is logically reversible). To put it in a different way. The difference between a logically reversible system and a (fully) reversible one is that the second can *itself* retrieve its past. For a logically reversible system, instead, retrieving its past is just possible, but it cannot be made by the system itself (we need to employ external resources to do it).

[13] Thus, while the *whole* dynamical structure of DS_2 is represented, through u, by corresponding structure of DS_1, the converse does not hold. This, from the mathematical point of view, just means that the emulation u is a *homomorphic* mapping of DS_2 in DS_1. The further fact that the state transition $h^1: a \rightarrow c$ corresponds, through u, to the state transition $g^2: u(a) \rightarrow u(c)$ (and not to $g^1: u(a) \rightarrow u(c)$, which does not exist) just means, from the mathematical point of view, that the emulation u is a *non-homologous* homomorphic mapping of DS_2 in DS_1.

[14] In addition, it is not clear *exactly how* we should intend such a strong requirement. Strictly speaking, even the original DS_2 fork $h^1: a \rightarrow c$, $h^1: b \rightarrow c$ does not satisfy it, for the two processes in this fork ar not *completely* distinct. In fact, they share a common subprocess, namely, the null or identical one $h^0: c \rightarrow c$.

[15] For example, the Galilean law of free fall is not exactly derivable from Newtonian mechanics and the law of universal gravitation.

[16] Churchland, at least once (1985, sec. 1), uses the expression "relevantly isomorphic" to refer to the mirroring relationship between S_2^* and S_2. Hooker refers to the same relationship as "the analog relation, A_R" (1981, 49). It has then become customary to refer to such relationship as "an isomorphism" or, sometimes, "a homomorphism" between S_2^* and S_2 (or conversely). This metaphorical usage of such terms should not be confused with their rigorous mathematical sense, which presupposes the formal definition of a specific kind of structure-preserving mapping.

[17] The reduction of *Classical Mechanics* (*CM*) to *Special Relativity Theory* (*SRT*) shows that a quite high faithfulness degree of the reduced theory image *CM** may result in a quite high law-mimicry value together with a *zero* retention value (complete ontological elimination). This happens because an accurate mimicry of *CM* by *CM** is obtained by replacing each *unary* property of *CM* (mass, length, time, etc.) with a *binary* relation of *CM** (mass with respect to a reference frame, etc.), so that identification between the corresponding attributes is hopeless (see Churchland 1979, sec. 11).

[18] Bickle's *New Wave Reduction* (1998, ch. 3) is a version of the Imaging Approach, in which (i) theories are construed as sets of models (semantically), rather than sets of sentences (syntactically), and thus (ii) reduction is not a deductive relationship between *formal* theories, but a relationship between sets of models that satisfies special conditions. Notwithstanding these differences, reduction is still analyzed by Bickle as a special relationship between theories (i.e. *sets* of models) and not as a representational relationship between *models*. Bickle shares his general view of reduction and theory structure with the *Structuralist Program* (Sneed 1971; Stegmüller 1976; Mayr 1976, 1981; Balzer, Pearce, and Smith 1984; Balzer, Moulines, and Sneed 1987).

[19] The term "isomorphic image" is intended here in its rigorous mathematical sense. This is not the sense in which the Imaging Approach employs the same term (see note 16). The general representational theory of reduction that I advocate can be thought as a development of *Suppes Reduction Paradigm* (1957, 271). The next section (sec. 7) will make clear that Schaffner's (1967) "too weak to be adequate" (Bickle 1998, ch. 3) criticism of Suppes Paradigm does not apply to my view.

[20] Since the functional description F typically contains several idealizations, no concrete or real system RS *exactly* satisfies F, but it rather fits F up to a certain degree. Thus, from a formal point of view, the application domain B_F of a phenomenon (F, B_F) might be better described as a fuzzy set.

[21] Hooker (2004, 436) maintains that "asymptotics provides the ground on which claims about inter-theoretic explanation, reduction and emergence must ultimately rest". According to him, "in physics, we find that the most famous theory pairs are all asymptotically related" (2004, 437). Among such pairs, he explicitly mentions: (i) special relativity and Newtonian mechanics; (ii) wave optics and ray optics; (iii) quantum mechanics and Newtonian mechanics; (iv) statistical mechanics and thermodynamics (where, in each pair, the first element is the *reducing* theory and the second element is the *reduced* theory). According to Hooker, an analogous relationship may also hold between two different models of the *same* theory; an example is the following pair of models of Newtonian mechanics: a non linear classic pendulum model and a harmonic oscillator model (2004, 438).

[22] In sec. 3 (see B1), I define a mathematical model MS as a set D together with a family $(\sigma_i)_{i\in I}$ of relations on D. This definition is fine as far as *relational* models are concerned, but not all mathematical models are of this kind. For instance, a topological space (with the standard axiomatization in terms of open sets) is not a relational model. Bourbaki 1968 (ch. 4) contains a quite general treatment of mathematical structures. However, Bourbaki's general theory of structures is developed at the metamathematical level. What we need is a theory of models developed *within* set theory, and thus at the mathematical level, as general as Bourbaki's metamathematical theory of structures.

[23] If $c = (m\pi\boldsymbol{l}, 0)$ for some $m \in Z$, then $g^t_{\boldsymbol{l}}(c) = c$ for any $t \in T$; that is to say, c is a fixed point of $DS_{n_{\boldsymbol{l}}}$. Furthermore, if m is not odd, for any $a \neq c$, for any $t \in T$, $g^t_{\boldsymbol{l}}(a) \neq c$. However, if m is odd, for some $a \neq c$ and some $t \in T$, $g^t_{\boldsymbol{l}}(a) = c$; therefore, the orbit of a is eventually periodic, and so $DS_{n_{\boldsymbol{l}}}$ is logically irreversible. It thus follows that the time set T of $DS_{n_{\boldsymbol{l}}}$ must be R^+. For, if it were R, $DS_{n_{\boldsymbol{l}}}$ would be reversible. As for the time set V of $DS_{o_{\boldsymbol{l}_0}}$, we may take it to be either R or R^+, for all state transitions $h^v_{\boldsymbol{l}_0}$ are both injective and surjective. If we take $V = R$, then $DS_{o_{\boldsymbol{l}_0}}$ is reversible. If we take $V = R^+$, then $DS_{o_{\boldsymbol{l}_0}}$ is irreversible but, since any $h^v_{\boldsymbol{l}_0}$ is surjective, $DS_{o_{\boldsymbol{l}_0}}$ has *complete past* and, since any $h^v_{\boldsymbol{l}_0}$ is injective, $DS_{o_{\boldsymbol{l}_0}}$ is *logically reversible*; that is to say, $DS_{o_{\boldsymbol{l}_0}}$ is *reversibly completable*. Here, it is convenient to take $V = R^+$, so that $DS_{o_{\boldsymbol{l}_0}}$ and $DS_{n_{\boldsymbol{l}}}$ have identical time set.

[24] According to the least upper bound property, for any non-empty subset A of R, if A has an upper bound, then the minimum of all upper bounds of A exists.

[25] A positive (negative) bob position x is the distance (the opposite of the distance), along the positive (negative) direction of the circular swinging path, of the bob itself from the intersection O between the path and the vertical straight line passing through the pendulum pivot. We take the positive path direction to be anticlockwise.

[26] See van Fraassen 1980 for a general discussion of the concept of an empirical substructure.

[27] Recall that, in this particular case, u is the identity function on $X\times Y$.

[28] Let I_H be an empirical interpretation of dynamical system DS on phenomenon H, and $\boldsymbol{M}_i$ $(1 \leq i \leq n)$ be any of the magnitudes of I_H; let Δ_i be the best accuracy degree of the measurements of magnitude $\boldsymbol{M}_i$. Then, *the best accuracy degree* Δ *of the measurements relative to* I_H is the minimum of all Δ_i, where $1 \leq i \leq n$.

[29] Notice that $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta_\Delta}}$ is reduced to $\boldsymbol{DS}_{n_{\boldsymbol{l}_0}}$, where $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta_\Delta}}$ is reversibly completable and $\boldsymbol{DS}_{n_{\boldsymbol{l}_0}}$ is not (see note 23). If we could also assert that $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta_\Delta}}$ is made up of $\boldsymbol{DS}_{n_{\boldsymbol{l}_0}}$, we would then conclude that being reversibly completable is an emergent property of $\boldsymbol{DS}_{o_{\boldsymbol{l}_0,\theta_\Delta}}$ relative to $\boldsymbol{DS}_{n_{\boldsymbol{l}_0}}$. Therefore, we would get an example of a pair of *empirically interpreted* dynamical systems such that (i) the first system is reduced to the second one and (ii) the first system has emergent properties relative to the second one. At the moment, however, we cannot reach this conclusion, for we have not stated any constitution condition for empirically interpreted dynamical systems.

[30] Thus, C_{F_2} is the set of states of DS_2 that corresponds to the *initial conditions* (of the abstract system AS_{F_2}) specified by the causal interaction scheme CS_{F_2}.

[31] Note that, in the pendulum models example, $\Delta_1 = \Delta_2 = \Delta$, for the magnitudes of the two interpretations $I_{H_{n_{\boldsymbol{l}_0}}} = I_{H_1}$ and $I_{H_{o_{\boldsymbol{l}_0,\theta}}} = I_{H_2}$ are the same; therefore, the condition $\Delta_1 \leq \Delta_2$ is obviously satisfied. In the general case, however, the two interpretations may contain different magnitudes, and so we must explicitly require that I_{H_1}'s measurements be at least as accurate as I_{H_2}'s.