Oriented Graphs and Dynamical Systems

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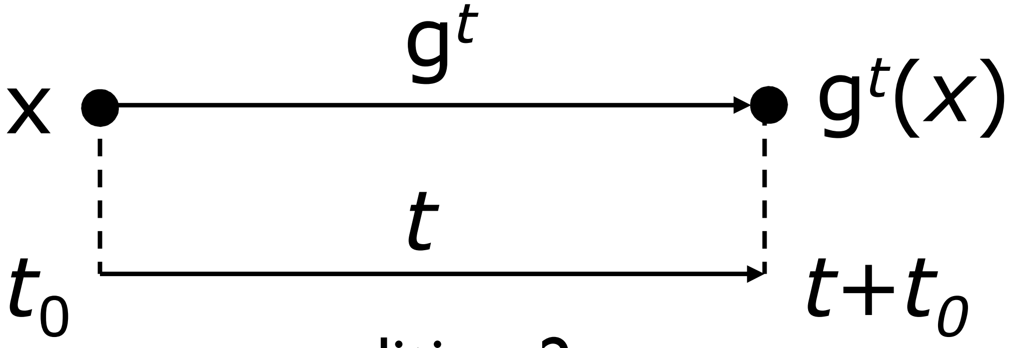
Dynamical systems — usual definition

- Let $T = Z^+$ (non-negative integers), Z (integers), R^+ (non-negative reals), or R (reals).

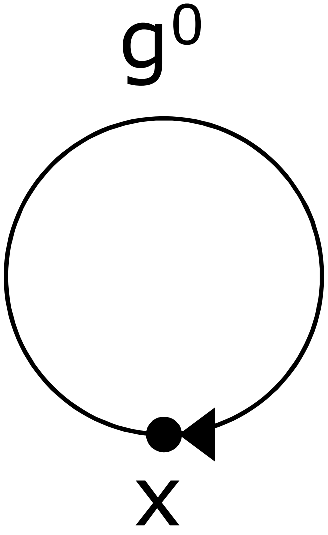
  DS *is a dynamical system* iff DS = (M, $(g^t)_{t\in T}$) and

1. M is a non-empty set; M represents all the possible states of the system, and it is called the *state space*;
2. $(g^t)_{t\in T}$ is a family of functions from M to M; each $t\in T$ is called a *duration*, and T is called the *time set* of DS; for any $t\in T$, g^t is called *the (state) transition of duration t*, or *the t-advance*, of DS;
3. for any $w, t \in T$, for any $x \in M$,
   1. $g^0(x) = x$;
   2. $g^{w+t}(x) = g^w(g^t(x))$.

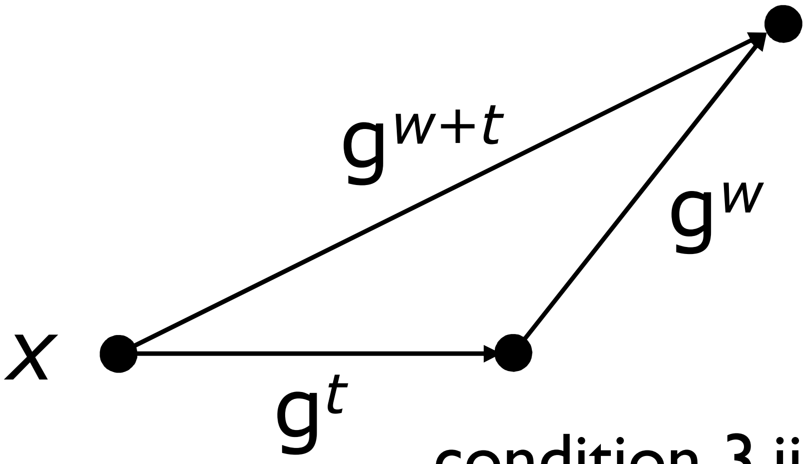
Intuitive meaning of the definition of a dynamical system



condition 2



condition 3.i



condition 3.ii

Looking at dynamics from the viewpoint of a minimal time-model

1. The previous definition is not completely general, for the essential properties (for the system's dynamics) of the time set are not made explicit.
    1. What is the *minimal* structure on the time set that allows us to define interesting dynamical concepts?
    2. Do their relationships depend on additional structure that the time set may support?
2. There is an interesting link between a dynamical system and
    1. the directed graph it induces on the state space;
    2. this graph and category theory.

- In what follows I will start from 2, and we will then see how this general perspective can shed some light on both 1.1 and 1.2.

Oriented graph (with identity and composition)

- An *oriented graph (with identity and composition)* is a pair (M, A), where M is a set of *points* (or *nodes*) and A is a set of *arrows* between points of M such that:

1. any two points in a given order (not necessarily distinct) are connected at most by one arrow;
   - hence, each arrow can be *identified* with the ordered pair whose first element is the departure (or source) point of the arrow and whose second element is its arrival (or target) point.
2. for any point, there is an arrow that departes and arrives to it;
   - hence, by 1, for any point x, there is *exactly* one arrow that connects x with itself, and the point x can thus be i*dentified* with this circular arrow; each such arrow is called an *identity arrow*;
3. for any two consecutive arrows (not necessarily distinct), there is always a third arrow (not necessarily distinct) that connects the departure point of the first arrow with the arrival point of the second one;
   - thus, by 1, this arrow is unique, and it is called *the composition* of the two consecutive arrows.

Oriented Graphs, Deductive Systems and Categories

- Note that any oriented graph can be made into a deductive system in Lambek's sense with the following definitions:
  - for any point x, the x-identity arrow id_x is $x \to x$
  - for any two consecutive arrows $x \to y$, $y \to v$, the composition operation, indicated by $(x \to y \to v)$, is defined by:
  - $(x \to y \to v) = x \to v$
- In addition, with the definitions above, any oriented graph is a category as well, for the following conditions hold:
  - for any three consecutive arrows $x \to y$, $y \to v$, $v \to z$,

    $((x \to y \to v) \to z) = (x \to v \to z) = x \to z$

    $(x \to (y \to v \to z)) = (x \to y \to z) = x \to z$
  - for any three arrows $x \to x$, $x \to y$, $y \to y$,

    $(x \to x \to y) = x \to y$ and $(x \to y \to y) = x \to y$

Labeled oriented graph

- A *labeled oriented graph* is an oriented graph where each arrow is assigned one or more labels.
- More rigorously, a labeled oriented graph G is defined as follows.
- G *is a labeled oriented graph* iff G = (M, A, *T*, f), (M, A) is an oriented graph, *T* is a non emtpy set (*T* is called *the set of labels of* G), and f is a function that, to any arrow member of A, associates a non-empty set of labels Z $\subseteq$ *T*.

Transition graph on a monoid L = (T, +)

- *G is a transition graph on L* iff G is a labeled oriented graph, L = (T, +) is a monoid, and the following conditions are satisfied:

1. T is the set of labels of G (any $t \in T$ is called a *duration*, T is called the *time set* of G, and L is called the *time model* of G;);
2. for any point, and for any duration $t \in T$, there is exactly one arrow with label t that departs from that point;
    - thus, in particular, there is no point from which two different arrows with the same label *depart*;
    - however, there may be points where (i) two different arrows with the same label *arrive*; (ii) two different arrows with *different* labels depart;
3. all identity arrows have label $0 \in T$, where 0 is the unity of L;
4. for any two consecutive arrows (not necessarily distinct), the first with label t and the second with label w, their composition arrow has label $w+t$.

Transition graphs, deductive systems and categories

- If G is a transition graph on L = (T, +), then G can naturally be made into a deductive system G^* in Lambek's sense. First, each arrow of the graph, together with each one of its labels, individuates a distinct arrow of the corresponding deductive system G^*; thus if $x \rightarrow x$ is an arrow of G, and t is one of its labels, then $x \ {}_{t}\!\!\longrightarrow x$ is an arrow of G^* and nothing else is an arrow of G^*; second, we define identity arrows and arrow composition as follows:
  - for any point x, the x-identity arrow id_x is $x \ {}_{0}\!\!\longrightarrow x$
  - for any two consecutive arrows $x \ {}_{t}\!\!\longrightarrow y$, $y \ {}_{w}\!\!\longrightarrow v$, the composition operation, indicated by $(x \ {}_{t}\!\!\longrightarrow y \ {}_{w}\!\!\longrightarrow v)$, is defined by:
  - $(x \ {}_{t}\!\!\longrightarrow y \ {}_{w}\!\!\longrightarrow v) = x \ {}_{w+t}\!\!\longrightarrow v$
- With the definitions above, the deductive system G^* turns out to be a category as well, for the following conditions hold:
  - for any three consecutive arrows $x \ {}_{u}\!\!\longrightarrow y$, $y \ {}_{t}\!\!\longrightarrow v$, $v \ {}_{w}\!\!\longrightarrow z$,

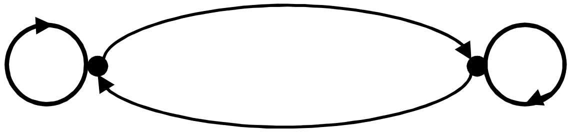
    $((x \ {}_{u}\!\!\longrightarrow y \ {}_{t}\!\!\longrightarrow v) \ {}_{w}\!\!\longrightarrow z) = (x \ {}_{t+u}\!\!\longrightarrow v \ {}_{w}\!\!\longrightarrow z) = x \ {}_{w+(t+u)}\!\!\longrightarrow z = x \ {}_{(w+t)+u}\!\!\longrightarrow z$

    $(x \ {}_{u}\!\!\longrightarrow (y \ {}_{t}\!\!\longrightarrow v \ {}_{w}\!\!\longrightarrow z)) = (x \ {}_{u}\!\!\longrightarrow x \ {}_{w+t}\!\!\longrightarrow z) = x \ {}_{(w+t)+u}\!\!\longrightarrow z$
  - for any three arrows $x \ {}_{0}\!\!\longrightarrow x$, $x \ {}_{t}\!\!\longrightarrow y$, $y \ {}_{0}\!\!\longrightarrow y$,

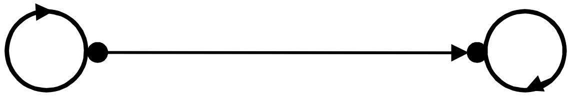
    $(x \ {}_{0}\!\!\longrightarrow x \ {}_{t}\!\!\longrightarrow y) = x \ {}_{t+0}\!\!\longrightarrow y = x \ {}_{t}\!\!\longrightarrow y$ and $(x \ {}_{t}\!\!\longrightarrow y \ {}_{0}\!\!\longrightarrow y) = x \ {}_{0+t}\!\!\longrightarrow y = x \ {}_{t}\!\!\longrightarrow y$

Invertible oriented graph

- An oriented graph G is *invertible* iff if x→y is an arrow of G, then y→x is an arrow of G.
- Obviously, by 1 of the definition of oriented graph, y→x is unique, and it is called the inverse of x→y.
- Below is an example of an invertible graph:



- Below is an example of a non-invertible graph:



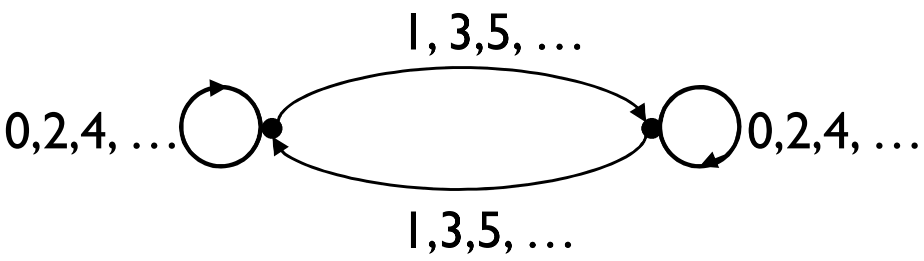
Group structure of L = (T, +) and invertibility of a transition graph G on L

Let G be a transition graph on L = (T, +);

- *Proposition 1.* If L is a group, then any arrow x → y of G has the inverse arrow y → x; in other words, if L is a group, then G is invertible.

Proof. Let *t* be one of the labels of x → y; let y → z be the arrow with label –t that departs from y; then (x → y → z) = x → z has label (–t)+t = 0; but x → x has label 0, and no two different arrows that depart from x have the same label; therefore z = x, and y → z = y → x is the inverse of x → y. Q.E.D.

- However, the converse is not true: If G is a transition graph on L and G is invertible, not necessarily is L a group. As a counter example, let G be the transition graph below, where L = (Z^+, +); G is invertible, but L is not a group.



Dynamical system on a monoid L = (T, +)

- *DS is a dynamical system on L* iff DS = (M, $(g^t)_{t \in T}$), L = (T, +) is a monoid, and the following conditions are satisfied:

1. the set M is not empty;
    - M is called the *state space* of DS;
2. $(g^t)_{t \in T}$ is a family, indexed by T, of functions from M to M;
    - each $t \in T$ is called a *duration*, T is called the *time set* of DS, and L is called the *time model* of DS;
    - for any $t \in T$, g^t is called *the (state) transition of duration t*, or *the t-advance*, of DS;
3. for any w, $t \in T$, for any x ∈ M,
    - $g^0(x) = x$, where 0 is the unity of L;
    - $g^{w+t}(x) = g^w(g^t(x))$.

The transition graph of a dynamical system

- Any dynamical system DS on a monoid L allows us to uniquely define a transition graph on L, as follows.
- G = (V, A, *U*, f) *is the transition graph of a dynamical system* DS = (M, $(g^t)_{t\in T}$) *on* L = (*T*, +) iff V = M, A = $\cup_{t\in T}\, g^t$, *U* = *T*, f: A → 2^T such that f(x, y) = {*t*: $g^t(x)$ = y}; the graph G is indicated by ***G***(DS, L).
- Note that, by the above definition,
  - *Proposition 2.* ***G***(DS, L) is a transition graph on L.

Proof. Details to be worked out.

The dynamical system of a transition graph

- Conversely, any transition graph G on a monoid L allows us to uniquely define a dynamical system on L, as follows.
- DS = (M, $(g^t)_{t\in T}$) *is the dynamical system of a transition graph* G = (V, A, *U*, f) *on* L = (*T*, +) iff M = V, *T* = *U*, for any $t \in T$, $g^t = \{a: a \in A \text{ and } t \in f(a)\}$; the system DS is indicated by ***DS***(G, L).
- Note that, by the above definition,
  - *Proposition* 3. ***DS***(G, L) is a dynamical system on L.

Proof. Details to be worked out.

For any L = (T, +), dynamical systems on L and transition graphs on L are equivalent theories

It is not difficult to show:

- *Proposition 4.* For any G, DS, L,
  1. if G is a transition graph on L, ***G***(***DS***(G, L), L) = G;
  2. if DS is a dynamical system on L, ***DS***(***G***(DS, L), L) = DS.

Proof. Details to be worked out.

- Proposition 4 thus ensures that, for any monoid L, the theory of dynamical systems on L and the theory of transition graphs on L are equivalent. (For (i) each model of the first theory is definable within exactly one model [*its definiens*] of the second one, (ii) conversely, and (iii) each model of either theory is the definiens of its definiens.)

Reversible vs. irreversible dynamical systems

- Let DS = (M, $(g^t)_{t \in T}$) be a dynamical system on a monoid L = (T, +).
- *DS is reversible* iff for any x, y $\in$ M, for any $t \in T$, if $g^t(x) = y$, there is $w \in T$ such that $g^w(y) = x$.
- *DS is irreversible* iff DS is not reversible.
- *Proposition* 5. DS is reversible iff the corresponding transition graph ***G***(DS, L) is invertible.

  Proof. Details to be worked out.
- *Conjecture* 6. If DS is reversible, then for any $t \in T$, g^t is a bijection.
  - Note that the converse does not hold. Countermodel: the dynamical system DS = (Z, $(s^k)_{k \in Z^+}$) on (Z^+, +), where s^0 is the identity function on Z and, for any k > 0, s^k is the k-th iteration of the successor function on Z.

  Disproof. Conjecture 6 is shown to be false by several countermodels. Among these, one which is a reversible, but not logically reversible dynamical system with incomplete past and finite state space (3 states), on a non-commutative and finite (4 durations) monoid. (For the definitions of logical reversibility and past incompleteness see dia 18.)

Future and past of a state (or set of states)

- Let DS = $(M, (g^t)_{t\in T})$ be a dynamical system on a monoid L = $(T, +)$; let $x \in X$.
- *$F^t(x)$ is the t-future of x* iff $t \in T-\{0\}$ and $F^t(x) = \{y: g^t(x) = y\}$.
  - $\forall x \in M, \forall t \in T-\{0\}, F^t(x)$ is a singleton.
- *$P^t(x)$ is the t-past of x* iff $t \in T-\{0\}$ and $P^t(x) = \{y: g^t(y) = x\}$.
- *F(x) is the future of x* iff $F(x) = \cup_{t\in T-\{0\}} F^t(x)$.
- *P(x) is the past of x* iff $P(x) = \cup_{t\in T-\{0\}} P^t(x)$.
- Analogous definitions can be given for a set of states $X \subseteq M$.

Different forms of reversibility/irreversibility

- Let DS = (M, $(g^t)_{t\in T}$) be a dynamical system on a monoid L = (*T*, +).
- *DS is logically reversible* iff $\forall t \in T$, g^t is injective.
  - DS is logically reversible iff $\forall x \in M$, $\forall t \in T-\{0\}$, $P^t(x)$ is a singleton.
- *DS is logically irreversible* iff $\exists t \in T$, g^t is not injective.
  - DS is logically irreversible iff $\exists x \in M$, $\exists t \in T-\{0\}$, $P^t(x)$ has at least two elements.
- *DS has complete past* iff $\forall t \in T$, g^t is surjective.
  - DS has complete past iff $\forall x \in M$, $\forall t \in T-\{0\}$, $P^t(x) \neq \varnothing$.
- *DS has incomplete past* iff $\exists t \in T$, g^t is not surjective.
  - DS has incomplete past iff $\exists x \in M$, $\exists t \in T-\{0\}$, $P^t(x) = \varnothing$.
- *DS is completely logically reversible* iff $\forall t \in T$, g^t is bijective.
- As conjecture 6 is false, reversibility and complete logical reversibility are logically independent concepts (neither one entails the other).
- However, Claudio Mazzola has shown that reversibility entails complete logical reversibility if the time model is commutative.

Time symmetry and time invertibility

- Let DS = $(M, (g^t)_{t \in T})$ be a dynamical system on a monoid L = $(T, +)$.
- *DS is time symmetric* iff DS is completely logically reversible and there is a function $-: M \to M$ such that $\forall t \in T, \forall x \in M$, $-(g^t(-(x))) = (g^t)^{-1}(x)$.
  - Note that, for any function – that satisfies the previous definition, $-(-(x)) = x$. (Hint: replace g^t by g^0.)
- *DS is time invertible* iff L is a group.
  - Note that, by proposition 1, any time invertible system is reversible.
  - We will see shortly that any time invertible system is completely logically reversible as well (proposition 10 below).
- Is the following true? (If yes, time invert. entails time symm.)
  - If DS is completely logically reversible, then DS is time symmetric.

Reversibility of a DS and gardens of Eden

- Let L = (T, +) be a monoid, and 0 be the unity of L; let DS = (M, $(g^t)_{t\in T}$) be a dynamical system on L.
- *x is a garden of Eden* iff $x\in M$ and, for any $y\in M$ and $t\in T$, if $t \neq 0$, then $g^t(y) \neq x$.
- *Proposition 7.* If DS is reversible, then DS has no garden of Eden.
  - Note that the converse does not hold. Countermodel: the dynamical system DS = (Z, $(s^k)_{k\in Z^+}$) on (Z^+, +), where s^0 is the identity function on Z and, for any $k > 0$, s^k is the k-th iteration of the successor function on Z.

Proof. Details to be worked out.

Motions, orbits and phase portrait of a DS

- Let DS = $(M, (g^t)_{t \in T})$ be a dynamical system on a monoid L = $(T, +)$.
- Let Y and Z be any two sets; let eval: $Z^Y \times Y \to Z$ such that, for any $h \in Z^Y$, for any $y \in Y$, eval(h, y) = h(y).
- *g_x is the motion (or the state evolution) with initial state* x of DS iff $g_x: T \to M$ and, for any $t \in T$, $g_x(t) = \text{eval}(g^t, x)$.
  - Intuitively, for any $x \in M$, the motion g_x represents the time evolution of the system when the state at the initial time t_0 is x.
  - N.B. The simpler definiens "for any $t \in T$, $g_x(t) = g^t(x)$" would not be formally correct, for in general g^t is a different function for different *t*s.
- For any $x \in M$, *the orbit of* x is the image of g_x; the orbit of x is indicated by orb(x).
  - Note that orb(x) = $\{z: z = g^t(x)$, for some $t \in T\}$.
- *r is an orbit* iff $\exists x \in M$ such that r = orb(x).
- *The phase portrait of* DS = {r: r is an orbit}.
  - That is, the phase portrait of DS is the set of all orbits.

The one-parameter monoid of a dynamical system on L = $(T, +)$

- Let DS = $(M, (g^t)_{t\in T})$ be a dynamical system on a monoid L = $(T, +)$, and 0 be the unity of L. Let us consider the set $\{h: h = g^t, \text{ for some } t\in T\} = \mathrm{Im}((g^t)_{t\in T})$; we indicate this set by $\{g^t\}_{t\in T}$. Let $\circ$ be the composition operation between functions from M to M.
  - Recall that, in general, a family $(f^j)_{j\in J}$ indexed by J of objects in X is a function $f: J \to X$; thus, $(f^j)_{j\in J} = f$ and $\mathrm{Im}((f^j)_{j\in J}) = \mathrm{Im}(f)$.
- *Proposition 8.* $(\{g^t\}_{t\in T}, \circ)$ is a monoid and its unity is g^0.
  - The monoid $(\{g^t\}_{t\in T}, \circ)$ is called *the one-parameter monoid of the dynamical system* DS on L.

Proof. Details to be worked out.

If the inverse $-t$ of duration t exists, then the corresponding t-advance g^t is bijective

- Let DS = (M, $(g^t)_{t\in T}$) be a dynamical system on a monoid L = (T, +), and 0 be the unity of L.
- As L is a monoid, for any $t\in T$, if an inverse of t exists, it is unique. In general, whenever the inverse of t exists, we indicate it by $-t$.
- *Proposition 9.* For any $t\in T$, if the inverse $-t$ of t exists, then

1. the inverse function $(g^t)^{-1}$ of g^t exists, and $(g^t)^{-1} = g^{-t}$;
2. g^t is a bijection;
3. the inverse of g^t with respect to the composition operation $\circ$ exists as well, and it is g^{-t}.

Proof. For any $x \in M$, $g^{-t}(g^t(x)) = g^{-t+t}(x) = g^0(x) = x$ and $g^t(g^{-t}(x)) = g^{t+(-t)}(x) = g^0(x) = x$; thus, g^{-t} is the inverse function of g^t. Furthermore, since g^{-t} is a function from M to M, g^t is a bijection. Finally, the inverse of g^t with respect to the composition operation is the inverse function $(g^t)^{-1} = g^{-t}$. Q.E.D.

The one-parameter group of a dynamical system on a group L = (T, +)

- Let DS = (M, $(g^t)_{t\in T}$) be a dynamical system on a group L = (T, +), and 0 be the unity of L. Let us consider the set $\{h: h = g^t$, for some $t\in T\}$ = Im($(g^t)_{t\in T}$); we indicate this set by $\{g^t\}_{t\in T}$. Let ∘ be the composition operation between functions from M to M.
- An immediate consequence of propositions 8 and 9 is the following:
- *Proposition* 10. If L is a group, ($\{g^t\}_{t\in T}$, ∘) is a group whose unity is g^0 and, for any $t\in T$,
  1. the inverse function $(g^t)^{-1}$ of g^t exists, and $(g^t)^{-1} = g^{-t}$;
  2. g^t is a bijection;
  3. the inverse of g^t with respect to the composition operation ∘ is g^{-t}.
  - The group ($\{g^t\}_{t\in T}$, ∘) is called *the one-parameter group of the dynamical system* DS on L.

Proof. From propositions 8 and 9.

Q.E.D.

Properties of (i) orbits of a DS on a monoid; (ii) orbits and motions of a DS on a group

- Let DS = (M, $(g^t)_{t\in T}$) be a dynamical system on a monoid L = (T, +).
- *Proposition 11.* For any x, y, z ∈ M, if z ∈ orb(x) and z ∈ orb(y), then orb(z) ⊆ orb(x) and orb(z) ⊆ orb(y);
  - that is to say, any two orbits that have a state in common both contain the whole orbit of that state; thus, there are no "crossing" orbits.

Proof. Details to be worked out.

- *Proposition 12.* If L is a group, then
  1. for any x, y, z ∈ M, if z ∈ orb(x) and z ∈ orb(y), then orb(x) = orb(y);
     - that is to say, for any z ∈ M, there is exactly one orbit that passes through z;
  2. for any x, y ∈ M, for any t ∈ T, if $g_x(t) = g_y(t)$, then $g_x = g_y$;
     - that is to say, if two state evolutions coincide at one instant, then they always coincide.

Proof. Details to be worked out.

Fixed points and periodic points

- Let DS = (M, $(g^t)_{t\in T}$) be a dynamical system on a monoid L = (T, +); let 0 be the unity of L.
- *x is a fixed point* iff orb(x) = {x}.
- *x is a periodic point* iff $\exists\, t \in T$ such that $t \neq 0$ and $g^t(x) = x$;
- *t is a period of x* iff $t \neq 0$ and $g^t(x) = x$.
  - Note that, by the definitions above, if t is a period of x, x is a periodic point; conversely, if x is a periodic point, there is t such that t is a period of x, but not necessarily is such a t unique. Also, by the definition above and th. 9, if t is a period of x and the inverse $-t$ of t exists, then $-t$ is a period of x as well.
- *t is the period of x* iff t is a period of x and, for any t^*, if t^* is a period of x, $t^* \neq$ t and $t^* \neq -t$, there is $k \geq 2$ such that $t^* = \underline{t+t\ldots}_{k \text{ times}}$ or $t^* = \underline{-t + -t\ \ldots}_{k \text{ times}}$.
  - Note that, by the above definition, if t is the period of x, then (i) t is unique iff the inverse of t does not exist, or $-t = t$; (ii) if t is not unique, then $-t$ is the period of x as well, but nothing else is the period of x.
- *Proposition* 13. If x is a fixed point, then x is a periodic point and, for any $t \in T$, if $t \neq 0$, t is a period of x.

  Proof. Details to be worked out.
- *Proposition* 14. If x is a fixed point, then

  1. if L = R^+ or R, then the period of x does not exist;
  2. if L = Z^+, then 1, and nothing else, is the period of x;
  3. if L = Z, then 1 is the period of x, -1 is the period of x, and nothing else is the period x.

  Proof. Details to be worked out.

Periodic, eventually periodic, aperiodic, and merging orbits

- Let DS = (M, $(g^t)_{t\in T}$) be a dynamical system on a monoid L = (T, +).
- *r is a periodic orbit* iff ∃x ∈ M such that r = orb(x) and x is a periodic point.
- *r is an eventually periodic orbit* iff r is an orbit, r is not a periodic orbit, and ∃y ∈ r such that orb(y) is periodic.
- *r is an aperiodic orbit* iff r is an orbit, r is not a periodic orbit, and r is not an eventually periodic orbit.
  - Note that (i) periodic, (ii) eventually periodic and (iii) aperiodic orbits form a partition of the phase portrait of DS.
- *r is a merging orbit* iff ∃x, y ∈ M such that r = orb(x), orb(x)∩orb(y) ≠ ∅, orb(x) ⊄ orb(y), and orb(y) ⊄ orb(x).
  - Note that (a) merging and (b) not merging orbits form an alternative partition of the phase portrait of DS.

Types of orbits in dynamical systems with special time models and finite state space

- *Proposition 15.* If DS = (M, $(g^t)_{t\in T}$) is a dynamical system on Z (R), and M is finite, then any orbit of DS is in the class
  1. not merging and periodic (b-i).

  Proof. Details to be worked out.
- *Proposition 16.* If DS = (M, $(g^t)_{t\in T}$) is a dynamical system on Z^+ (R^+), and M is finite, then any orbit of DS is either
  1. not merging and periodic (b-i);
  2. not merging and eventually periodic (b-ii);
  3. merging and eventually periodic (a-ii);

  and, for any of the previous classes, there is DS = (M, $(g^t)_{t\in T}$) on Z^+ (R^+) with M finite and some orbit in that class.

  Proof. Details to be worked out.

Types of orbits in dynamical systems with special time models and infinite state space

- *Proposition 17.* If DS = $(M, (g^t)_{t\in T})$ is a dynamical system on Z (R), and M is not finite, then any orbit of DS is either
  1. not merging and periodic (b-i);
  2. not merging and aperiodic (b-iii);

  and, for any of the previous classes, there is DS on Z (R) with M not finite and some orbit in that class.

  Proof. Details to be worked out.
- *Proposition 18.* If DS = $(M, (g^t)_{t\in T})$ is a dynamical system on Z^+ (R^+), and M is not finite, then any orbit of DS is either
  1. not merging and periodic (b-i);
  2. not merging and aperiodic (b-iii);
  3. not merging and eventually periodic (b-ii);
  4. merging and aperiodic (a-iii);
  5. merging and eventually periodic (a-ii);

  and, for any of the previous classes, there is DS on Z^+ (R^+) with M not finite and some orbit in that class.

  Proof. Details to be worked out.