Reduction in Dynamical Systems: A Representational View

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1 Introduction

Standard accounts have traditionally viewed reduction as a *deductive* relationship between two *formal* theories [23]. Schaffner's *General Reduction Paradigm* [25] was an early attempt to modify Nagel's classic account, so as to accomodate cases where the reduced theory is, strictly speaking, false. The most comprehensive and detailed deductivist account of reduction is Churchland and Hooker's *Imaging Approach* [9, 10, 13, 14, 16], which can be seen as a creative development of Nagel's basic insights, as well as a sensible departure from Nagel's explicit tenets [4, 6, 7, 19]. [19] has convincingly argued that Kim's *Functionalizing Approach* to reduction [17] is in fact a version of Nagel's account; such a version is essentially equivalent to the Imaging Approach.

This paper proposes an alternative view, according to which reduction is better conceived as a *representational* relationship between two mathematical *models* MS_1 and MS_2, which grants the retrieval, within the representing model MS_1, of an isomorphic image of MS_2.[1]

Bickle's *New Wave Reduction* [6, ch. 3] is a version of the Imaging Approach by Churchland and Hooker in which (i) theories are construed as sets of models (semantically), rather than sets of sentences (syntactically), and thus (ii) reduction is not a *deductive* relationship between *formal* theories, but a relationship between *semantic* theories (i.e. *sets* of models) that satisfies special conditions. Notwithstanding these differences, reduction is still analyzed by Bickle as a special relationship between *theories* (i.e. *sets* of models) and not as a representational relationship between *models*. Bickle shares his general view of reduction and theory structure with the *Structuralist Program* [27, 28, 21, 22, 3, 2].

The general representational theory of reduction that I advocate is in broad agreement with Suppes' *Reduction Paradigm* [29, 271],[2] and it is somehow consonant with some of the ideas of Hooker's *dynamically based* revision of the Imaging Approach [15].

Compared to traditional approaches to reduction (*deductivist* or, more generally, *theory-based* approaches), the representational one has several advantages, whose details will only be apparent later. For the moment, it suffice to

[1] The term "isomorphic image" is intended here in its rigorous mathematical sense. This is not the sense in which the Imaging Approach employs the same term.

[2] Section 4 (see case 2) will make clear that Schaffner's [25] "too weak to be adequate" [6, ch. 3] criticism of Suppes' Reduction Paradigm does not apply to my view.

say that the representational theory scores better as far as precision and depth of analysis are concerned. Also, this theory is fostering a unified, conceptually crisp, and formally developed account of *prima facie* conflicting aspects of reduction – total and exact reduction *vs.* partial, approximate and asymptotic one, on which traditional approaches hardly fare as well.

I will develop this general representational theory only for the special case of dynamical systems. As intended here [1, 30, 12], a *dynamical system* is a kind of mathematical model that captures the intuitive idea of an arbitrary deterministic system. Models of this kind allow us to study in a precise way typical features of complex systems. Among them, in recent years, the one of emulation has gained growing attention [33, 34, 35, 36, 37]. Intuitively, a dynamical system DS_1 *emulates* a second dynamical system DS_2 when the first one exactly reproduces the whole dynamics of the second one.

The emulation relationship can be defined in a precise way for any two arbitrary dynamical systems and it has been shown [12, ch. 1, th. 11] that, if DS_1 emulates DS_2, there is a third system DS_3 such that (i) DS_2 is isomorphic to DS_3; (ii) all states of DS_3 are states of DS_1; (iii) any state transition of DS_3 is constructed out of state transitions of DS_1. In this paper, I will focus on a more general version of this theorem [*Virtual System Theorem VST*], which is based on a weaker and simpler definition of emulation. I will then argue that this result allows us to claim: If DS_1 emulates DS_2, then DS_2 is *reduced* to DS_1.

The claim that emulation is sufficient for reduction (in force of [*VST*]) is a precise statement of the representational view of reduction for the special case of dynamical systems. Strictly speaking, this claim is intended to hold exclusively for dynamical systems as purely *mathematical models* with no empirical interpretation. In a different sense, however, dynamical systems typically function as *models of real phenomena*. In this second sense, a dynamical system is not a purely mathematical entity *DS*, but it is a pair (DS, I_H), where I_H is an empirical interpretation that links the purely mathematical model *DS* to a phenomenon *H*. This paper will also provide the main lines of an extension of the representational theory of reduction to *empirically interpreted* dynamical systems.

As said, the emulation relationship is the basis of a representational view of reduction for dynamical systems (either empirically interpreted or not). The simplest form of such relationship holds between two dynamical systems DS_1 and DS_2 when the *whole* dynamics of DS_2 is *exactly* reproduced by DS_1. This simple form may very well be the basis for a representational account of *total* and *exact* reduction, but we need a more sophisticated version of emulation for dealing with cases of asymptotic, partial and approximate reduction [15]. Such a version will be introduced in section 5, where it will then be employed for a treatment of partial and approximate reduction in empirically interpreted dynamical systems.

2 Dynamical systems and emulation

A dynamical system is a kind of mathematical model that formally expresses the notion of an arbitrary deterministic system, either reversible or irreversible, with discrete or continuous time or state space. Let Z be the integers, Z^+ the non-negative integers, R the reals and R^+ the non-negative reals; below is the exact definition of a dynamical system.

[1] *DS is a dynamical system* iff DS is a pair $(M, (g^t)_{t \in T})$ such that

1. M is a non-empty set; M represents all the possible states of the system, and it is called the *state space*;
2. T is either Z, Z^+, R, or R^+; T represents the time of the system, and it is called the *time set*; any $t \in T$ is called a *duration* of the system;
3. $(g^t)_{t \in T}$ is a family of functions from M to M; each function g^t is called a *state transition* of duration t, or a *t-advance*, of the system;
4. for any t, $v \in T$, for any $x \in M$, $g^0(x) = x$ and $g^{t+v}(x) = g^v(g^t(x))$.

[2] A *discrete dynamical system* is a dynamical system whose state space is finite or denumerable, and whose time set is either Z or Z^+; examples of discrete dynamical systems are Turing machines and cellular automata.[3]

[3] A *continuous dynamical system* is a dynamical system that is not discrete; examples of continuous dynamical systems are iterated mappings on R, and systems specified by ordinary differential equations.

[4] A *possible dynamical system* is a pair $(M, (g^t)_{t \in T})$ that satisfies the first three conditions of definition [1].

We can now define the concept of an isomorphism between two possible dynamical systems as follows.

[5] *r is an isomorphism of DS_1 in DS_2* iff $DS_1 = (M, (g^t)_{t \in T})$ and $DS_2 = (N, (h^v)_{v \in V})$ are possible dynamical systems, $T = V$, r: $M \rightarrow N$ is a bijection and, for any $t \in T$, for any $x \in M$, $r(g^t(x)) = h^t(r(x))$.

[6] *DS_1 is isomorphic* to DS_2 iff there is r such that r is an isomorphism of DS_1 in DS_2.

It is easy to verify that the isomorphism relation is an equivalence relation on any given set of possible dynamical systems. (The concept of *set of all possible dynamical systems* is inconsistent, and we must then take as the basis of the theory of dynamical systems a specific, sufficiently large, set of possible dynamical systems.) It is also not difficult to prove that the relation of isomorphism is compatible with the property of being a dynamical system, that is to say: if DS_1 is isomorphic to DS_2 and DS_1 is a dynamical system, then DS_2 is a dynamical system. This allows us to speak of abstract dynamical systems in exactly the same sense we talk of abstract groups, fields, lattices, order structures, etc. We can thus define:

[7] an *abstract dynamical system* is any equivalence class of isomorphic dynamical systems.

It is easily shown that any two dynamical systems have exactly the same *structural properties* iff they are isomorphic.[4] Since *general dynamical sys-*

[3]The term "discrete dynamical system" is often used (see, for example, [18, 20, 24]) as a synonym for "dynamical system with discrete time", i.e., according to [30], a *cascade.* My use of the term "discrete dynamical system" is in accordance with [31].

[4]P is a *structural property of a dynamical system* (or a *dynamical property*) iff for any two mathematical models MS_1 and MS_2, (i) if MS_1 has P, MS_1 is a dynamical system and (ii) if MS_1 has P, and MS_1 is isomorphic to MS_2, then MS_2 has P. Thus, a dynamical property is a property *specific* to dynamical systems that is *preserved* by isomorphism.

tems theory[5] is exclusively interested in such properties, it regards any two isomorphic systems as identical.

Dynamical systems are appropriate models to study several interesting features of complex systems. The one of emulation is typical of computational systems [37], but it can in principle involve any two dynamical systems. The intuitive idea is that a dynamical system DS_1 emulates a second dynamical system DS_2 when the first one exactly reproduces the whole dynamics of the second one. Here are some examples. A universal Turing machine emulates any Turing machine; for any Turing machine TM there is a cellular automaton CA such that CA emulates TM [26, th. 3], and vice versa; the simple cellular automaton specified by Wolfram's rule 18 emulates the one specified by rule 90 (both CA are monodimensional, with 2 possible values for cell, and neighborhood of radius 1; see [34, p. 20]).

[12, ch. 1, def. 4] gave a formal definition of the emulation relationship that applies to any two arbitrary dynamical systems. Here, I will employ a weaker and simpler definition (see figure 1), which nevertheless suffices for the present purposes.

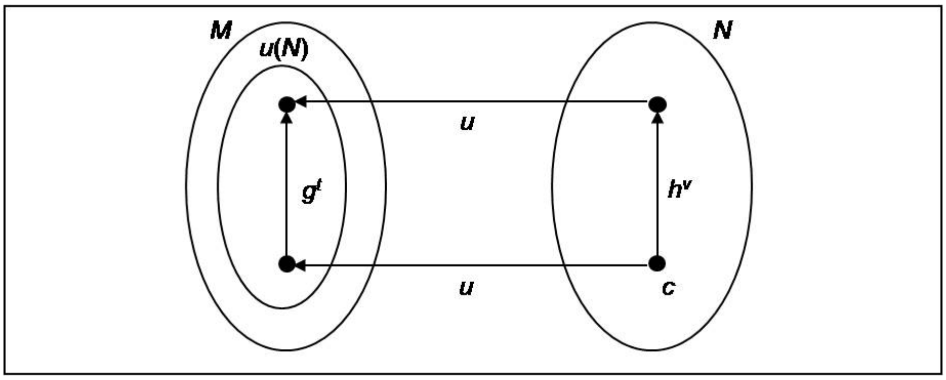

Figure 1. Emulation

[8] DS_1 *emulates* DS_2 iff $DS_1 = (M, (g^t)_{t \in T})$ and $DS_2 = (N, (h^v)_{v \in V})$ are dynamical systems, and there is an injective function u: $N \to M$ such that, for any $v \in V$, for any $c \in N$, there is $t \in T$ such that $u(h^v(c)) = g^t(u(c))$. Any function u that satisfies the previous condition is called an *emulation of* DS_2 *in* DS_1.

3 Emulation is sufficient for reduction

[12, ch 1, th. 11] proved that, if u is an emulation of DS_2 in DS_1, there is a third system DS_3 such that (i) u is an isomorphism of DS_2 in DS_3; (ii) all

The proof that any two isomorphic dynamical systems have exactly the same dynamical properties is immediate. Conversely, for any two non-isomorphic dynamical systems DS_1 and DS_2, there is a dynamical property they do not share; namely, the property of being isomorphic to DS_1.

[5] By *general dynamical systems theory* I mean the mathematical theory whose Suppes' style axiomatization [29, ch. 12] is given by def. [1].

states of DS_3 are states of DS_1; (iii) any state transition of DS_3 is constructed out of state transitions of DS_1. This result still holds for the weaker definition of emulation [8], as the following theorem shows.

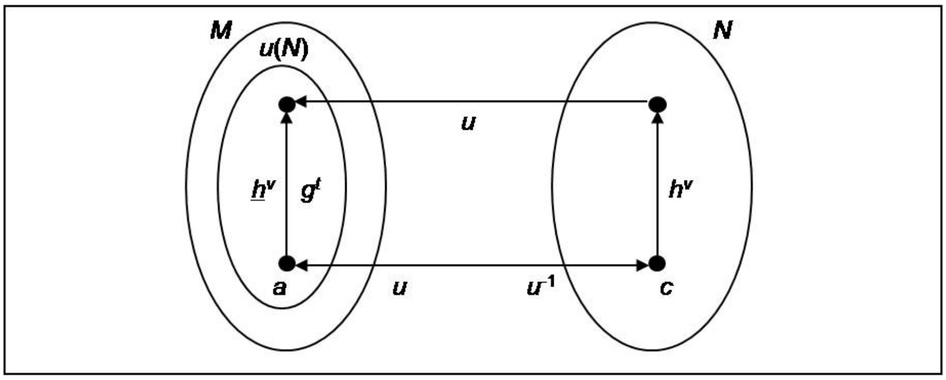

Figure 2. The u-virtual system DS_2 in DS_1

Virtual System Theorem $[VST]$

• Let $DS_1 = (M, (g^t)_{t \in T})$ and $DS_2 = (N, (h^v)_{v \in V})$ be dynamical systems, and u be an emulation of DS_2 *in* DS_1;

• let $DS_3 = (\underline{N}, (\underline{h}^v)_{v \in V})$, where $\underline{N} = u(N)$ and, for any $a \in \underline{N}$, for any $v \in V$, $\underline{h}^v(a) = u(h^v(u^{-1}(a))$; the system DS_3 is called *the u-virtual system* DS_2 *in* DS_1 (see figure 2);

then:

(i)u is an isomorphism of DS_2 in DS_3;

(ii) all states of DS_3 are states of DS_1;

(iii) for any state transition $\underline{h}^v$ of DS_3, for any $a \in \underline{N}$, there is a state transition g^t of DS_1 such that $\underline{h}^v(a) = g^t(a)$.

Proof of (i)

By the definition of DS_3, for any $c \in N$, $u(h^v(c)) = u(h^v(u^{-1}(u(c))) = \underline{h}^v(u(c))$. Therefore, by the definition of isomorphism [5], u is an isomorphism of DS_2 in DS_3.

Proof of (ii)

Obvious, by the definition of DS_3.

Proof of (iii)

By the definition of DS_3, for any $v \in V$, for any $a \in \underline{N}$, $\underline{h}^v(a) = u(h^v(u^{-1}(a))$. Let $c = u^{-1}(a)$. Since u is an emulation of DS_2 in DS_1, by definition [8], there is $t \in T$ such that $u(h^v(c)) = g^t(u(c))$. Therefore, $\underline{h}^v(a) = g^t(u(c)) = g^t(a)$. Q.E.D.

It is my contention that, if a dynamical system DS_1 emulates a second system DS_2, $[VST]$ allows us to claim that DS_2 *is reduced to* DS_1. In other words, I maintain that, because of $[VST]$, emulation is sufficient for *reduction*.

Before seeing the details of the supporting argument, it is important to make clear that dynamical systems, as intended here, are *purely mathematical* entities with no empirical interpretation; that is to say, at this level of analysis, a dynamical system is just a model of the mathematical theory whose Suppes' style axiomatization [29, ch. 12] is given by def. [1]. The claim that emulation is sufficient for reduction is thus exclusively limited to dynamical systems intended in this sense.

As just said, when I speak of a dynamical system as a *model*, I mean a model of a quite general *mathematical theory*, whose axiomatization is expressed by the definition, in set theory, of an appropriate set-theoretical predicate (def. [1]). It is important to sharply distinguish this sense of the term "model" from a different one, which also applies to dynamical systems, and is equally central to a complete understanding of their epistemological status. This second sense is the one intended when we say that a specific dynamical system is a model of a *real phenomenon*; however, this sense does not refer to a dynamical system as a purely mathematical entity (i.e., just a model of general dynamical system theory) but, rather, to such entity *together with* an empirical interpretation that links the mathematical model to the phenomenon which it is intended to describe.

A simple example will make the distinction clear. Let us consider the following system of two ordinary differential equations $\langle dy(v)/dv = \dot{y}(v), d\dot{y}(v)/dv = -\boldsymbol{g}\rangle$, where $\boldsymbol{g}$ is a fixed real positive constant. The solutions of such equations uniquely determine the dynamical system $DS_e = (Y \times \dot{Y}, (h^v)_{v \in V})$, where $Y = \dot{Y} = V = R$ (the real numbers) and, for any v, y, $\dot{y} \in R$, $h^v(y, \dot{y}) = (-\boldsymbol{g}v^2/2 + \dot{y}v + y, -\boldsymbol{g}v + \dot{y})$. It is immediate to verify that DS_e satisfies def. [1], so that it is a model in the first sense.

On the other hand, let us consider the phenomenon of the free fall of a medium size body in the vicinity of the earth (henceforth, H_e), and let us interpret the first component Y of the state space of DS_e as the set of all possible values of the *vertical position* of an arbitrary free falling body, the second component $\dot{Y}$ as the set of all possible values of the *vertical velocity* of the falling body,[6] and the time set V of DS_e as the set of all possible values of *physical time*. Since all three of these magnitudes are measurable or detectable properties of the intended phenomenon H_e, the given interpretation is an *empirical* interpretation of the dynamical system DS_e on H_e. Let I_{H_e} be such an interpretation. Then, the pair $(DS_e, I_{H_e}) = \boldsymbol{DS}_e$ is an empirical model of H_e, i.e., such a *pair* is a model in the second sense. $\boldsymbol{DS}_e$ will be called the *falling body model*.

My claim that emulation is sufficient for reduction (in force of $[VST]$) is intended to hold exclusively for dynamical models in the first sense. This does not mean that such a claim does not have any bearing on the further question: What are the conditions for reduction of an empirically interpreted dynamical system (DS_2, I_2) to another one (DS_1, I_1)? I will return later (see sec. 4) to this question. For the moment, it suffice to say that, in my view, the conditions for reduction of the mathematical model DS_2 to the mathematical model DS_1 are a necessary component of the more complex conditions for

[6] For any falling body a, if p_a is the point where a is initially released, a's *vertical* position and velocity are taken with respect to an axis with origin in the earth center that passes through p_a; the positive direction of such axis is from the earth center to its surface.

reduction of (DS_2, I_2) to (DS_1, I_1).

I am now going to present a detailed argument to support the claim that emulation is sufficient for reduction. The complete argument relies on five premises, divided into three groups. The first premise (**A**) is the most general one, for it refers to systems of *any* kind. Specifically, **A** states a sufficient condition for reduction between two arbitrary systems. The premises of the second group (**B1** and **B2**) are at an itermediate level of generality, for they refer exclusively to *mathematical* systems of any kind, that is, systems that are models of *some* mathematical theory. **B1** explicitly states what it is to be intended for "constitutive entity of a mathematical model", while **B2** makes clear the meaning of "whole structure of a mathematical model". The premises of the third group (**C1** and **C2**) are the most specific, for they refer to *dynamical systems* (in the purely mathematical sense). In particular, **C1** states identity conditions for such systems, and **C2** makes explicit the exact meaning of "whole structure of a dynamical system". Below are the five premises. Each of them is followed by a brief elucidation, which is intended to pin point crucial features of the corresponding premise, as well as to provide an intuitive justification for its assumption.

A For a system S_2 to be reduced to a system S_1, it is sufficient that (a) all the constitutive entities of S_2 are constitutive entities of S_1 and (b) the whole structure of S_2 is a part of the whole structure of S_1. *Elucidation* – In general, a system S is characterized by a whole structure formed by a complex of interconnected elements; each of these structural elements is built out of a given stock of building blocks, which we call "the constitutive entities of S". Thus, if two systems S_1 and S_2 satisfy conditions (a) and (b) above, the system S_2 is in fact a subsystem of S_1; this allows us to claim that S_2 is reduced to S_1.

B1 The constitutive entities of a mathematical model are the entities in its domain. *Elucidation* – According to standard definition, a mathematical model MS is a set D together with a family $(\sigma_i)_{i \in I}$ of relations on D. For any $i \in I$, there is exactly one $n \geq 0$ such that σ_i has arity n, where relations of arity 0 are identified with members of D, and relations of arity $n > 0$ are identified with sets of n-tuples of members of D; the set D is called the *domain* of the model. A mathematical model can thus be thought as a special kind of system, whose structural elements are the relations in the family $(\sigma_i)_{i \in I}$, and whose constitutive entities are the members of D.

B2 The *whole* structure of a mathematical model $MS = (D, (\sigma_i)_{i \in I})$ is the union of all the relations in the family $(\sigma_i)_{i \in I}$;[7] accordingly, if the relata of "is a part of" are whole structures of mathematical models, "is a part of" is to be interpreted as set-inclusion. *Elucidation* – We have just seen that a mathematical model can be thought as a special kind of system, whose structural elements are the relations in the family $(\sigma_i)_{i \in I}$. Each of such relations is a set of n-tuples; thus, the union of these sets is the whole structure

[7]The condition in the text holds iff any relation σ_i has arity > 0. The general condition is as follows. Let $X = \{x$: for some $i \in I$, $x = \sigma_i$ and σ_i is a relation of arity $0\}$; then, the whole structure of $(D, (\sigma_i)_{i \in I})$ is the union of X and all relations σ_i of arity > 0. Obviously, this condition reduces to the one in the text when X is empty, i.e., when any relation σ_i has arity > 0.

formed by the complex of such relations. Given this interpretation of "whole structure of a mathematical model", it is then obvious that "is a part of" should be interpreted as set-inclusion.

C1 From the point of view of general dynamical systems theory, any two isomorphic dynamical systems are identical. *Elucidation* – General dynamical systems theory studies the structural properties (see notes 4 and 5) of dynamical systems, and any two dynamical systems have exactly the same structural properties iff they are isomorphic. Therefore, general dynamical systems theory does not distinguish between any two isomorphic dynamical systems.

C2 If a mathematical model is a dynamical system $DS = (M, (g^t)_{t \in T})$, the whole structure of the model is the set of all state pairs (x, y) such that, for some $t \in T$, $g^t(x) = y$. *Elucidation* – We should first of all notice that, by def. [1], a dynamical system is a mathematical model of a special kind, namely, such that any relation g^t is in fact a function from M to M. Then, **C2** is an immediate consequence of this observation and **B2**.[8]

Sufficiency of emulation for reduction

1. For a mathematical model MS_2 to be reduced to a mathematical model MS_1, it is sufficient that (a) the domain of MS_2 is included in the domain of MS_1 and (b) the whole structure of MS_2 is included in the whole structure of MS_1; (logically follows from **A**, **B1** and **B2**;)
2. hence, if u is an emulation of DS_2 in DS_1, the u-virtual system DS_2 in DS_1 is reduced to DS_1; (logically follows from 1, **C2**, and theses (ii) and (iii) of [VST];)
3. if u is an emulation of DS_2 in DS_1, DS_2 is isomorphic to the u-virtual system DS_2 in DS_1; (logically follows from thesis (i) of [VST] and def. [6];)
4. consequently, if u is an emulation of DS_2 in DS_1, DS_2 is reduced to DS_1. (Logically follows from 2, 3, **C1** and the fact that dynamical systems, as intended here, are just models of general dynamical systems theory.)

4 Models of phenomena—sufficient conditions for total and exact reduction in empirically interpreted dynamical systems

Thus far, the *representational theory* of reduction has a precise formulation only if the models involved are dynamical systems in the purely mathematical sense. However, we have seen in sec. 3 that dynamical systems can also be intended as *models of real phenomena.* According to this second sense of the term "model", a dynamical system is not a purely mathematical entity DS; rather, it is a pair (DS, I_H), where I_H is an empirical interpretation that links the purely mathematical model DS to a phenomenon H. The representational theory should then be further developed to provide conditions for reduction of an empirically interpreted dynamical system (DS_2, I_{H_2}) to another one (DS_1, I_{H_1}). I will briefly sketch here the main lines of such development. The following exposition has no pretention to exhaustiveness. Its goal is just

[8] Thus, **C2** is not an *independent* premise of the argument, for it is entailed by def. [1], the standard definition of a mathematical model, and **B2**.

to trace a possible way along which an adequate representational theory of reduction for *empirically interpreted* dynamical systems might be worked out.

In general, a *phenomenon* H can be thought as a pair $(F,\, B_F)$ of two distinct elements. The first one, F, is a *functional description* of (i) an abstract type of real system AS_F and (ii) a general spatio-temporal scheme CS_F of its causal interactions; in particular, the functional description of the abstract system AS_F specifies its structural elements (or functional parts) and their mutual relationships and organization, while the description of the causal scheme CS_F specifies the initial conditions of AS_F's evolution. The second element, B_F, is the set of all concrete systems of type AS_F that also satisfy the causal interaction scheme CS_F; B_F is called the *application domain*[9] of the phenomenon H.

For example, let $H_e = (F_e,\, B_{F_e})$ be the phenomenon of the free fall of a medium size body in the vicinity of the earth (from now on, I will refer to H_e just as *the phenomenon of free fall*). In this case, the functional description F_e is as follows. The abstract type of real system AS_{F_e} has just one structural element, namely, a medium size body in the vicinity of the earth; the causal interaction scheme CS_{F_e} consists in releasing the body at an arbitrary instant, and with a *purely vertical* velocity and position (relative to the earth surface) whose respective values are within appropriate boundaries. B_{F_e} is then the set of all concrete medium size bodies in the vicinity of the earth that satisfy the given scheme of causal interactions. Any such body will be called a (*free*) *falling body*.

Let $DS = (X_1 \times \ldots \times X_n,\, (g^t)_{t \in T})$ be a dynamical system whose state space $M = X_1 \times \ldots \times X_n$ has n components X_i ($1 \le i \le n$, where $i,\, n \in Z^+ =$ the non negative integers). An *interpretation* I_H *of* DS *on a phenomenon* H consists in identifying each component X_i with the set of all possible values of a magnitude $\boldsymbol{M}_i$ of the phenomenon H, and the time set T with the set of all possible instants of the time $\boldsymbol{T}$ of H itself. An interpretation I_H of DS on H is *empirical* if the time $\boldsymbol{T}$ and all the magnitudes $\boldsymbol{M}_i$ are measurable properties of the phenomenon H. A pair $(DS,\, I_H)$, where DS is a dynamical system with n components and I_H is an interpretation of DS on H, is said to be a *model of the phenomenon* H. If the interpretation I_H is empirical, then $(DS,\, I_H)$ is an *empirical model of* H. Such a model is said to be *empirically correct* if, for any i, all measurements of magnitude $\boldsymbol{M}_i$ are consistent with the corresponding values x_i determined by DS. An empirically correct model of H is also called a *Galilean model of* H [11], [12, ch. 3]. A *Galilean model* is then any empirically correct model of some phenomenon.

As an example, let us consider again the phenomenon of free fall H_e. Let DS_e be the dynamical system with two components specified in sec. 3, and I_{H_e} be its interpretation given in sec. 3; then, according to the previous definitions, I_{H_e} is an empirical interpretation of DS_e on H_e, and $(DS_e,\, I_{H_e}) = \boldsymbol{DS}_e$ is an empirical model of H_e. For an appropriate value of the constant

[9] Since the functional description F typically contains several idealizations, no concrete or real system RS *exactly* satisfies F, but it rather fits F up to a certain degree. Thus, from a formal point of view, the application domain B_F of a phenomenon $(F,\, B_F)$ might be better described as a fuzzy set.

$\boldsymbol{g}$, such a model also turns out to be empirically correct.[10]

Let us now consider two phenomena $H_1 = (F_1, B_{F_1})$ and $H_2 = (F_2, B_{F_2})$, and two empirically interpreted dynamical systems $\boldsymbol{DS}_1 = (DS_1, I_{H_1})$ and $\boldsymbol{DS}_2 = (DS_2, I_{H_2})$ such that $\boldsymbol{DS}_1$ is an empirical model of H_1 and $\boldsymbol{DS}_2$ is an empirical model of H_2. What are the conditions for reduction of $\boldsymbol{DS}_2$ to $\boldsymbol{DS}_1$? I will divide the discussion into three distinct cases.

Case 1. Let us suppose that $B_{F_2} \subseteq B_{F_1}$. Under this hypothesis, it seems sensible to claim that, if DS_1 emulates DS_2, then $\boldsymbol{DS}_2$ *is reduced to* $\boldsymbol{DS}_1$. To see this point, let us notice, first, that the hypothesis $B_{F_2} \subseteq B_{F_1}$ ensures that any concrete system described by $\boldsymbol{DS}_2$ is also described by $\boldsymbol{DS}_1$. Second, let u: $Y_1 \times \ldots \times Y_n \to X_1 \times \ldots \times X_m$ be an emulation of $DS_2 = (Y_1 \times \ldots \times Y_n, (h^v)_{v \in V})$ in $DS_1 = (X_1 \times \ldots \times X_m, (g^t)_{t \in T})$. Thus, by def. [8], any state transition h^v: $(y_1, \ldots, y_n) \to (y_1', \ldots, y_n')$ corresponds to a state transition g^t: $(x_1, \ldots, x_m) \to (x_1', \ldots, x_m')$, where $u(y_1, \ldots, y_n) = (x_1, \ldots, x_m)$ and $u(y_1', \ldots, y_n') = (x_1', \ldots, x_m')$. In addition, since $\boldsymbol{DS}_2$ is a model of H_2, for any j, y_j and y_j' are values of a measurable magnitude $\boldsymbol{M}_j$ of H_2, and v is a value of the time $\boldsymbol{T}_2$ of H_2; on the other hand, since $\boldsymbol{DS}_1$ is an empirical model of H_1, for any i, x_i and x_i' are values of a measurable magnitude $\boldsymbol{M}_i$ of H_1, and t is a value of the time $\boldsymbol{T}_1$ of H_1. For any concrete system $RS \in B_{F_2}$, both the $\boldsymbol{DS}_2$ and the $\boldsymbol{DS}_1$ descriptions apply to RS. But then, the emulation function u tells us exactly how the $\boldsymbol{DS}_2$ description of RS corresponds to the $\boldsymbol{DS}_1$ description.

As an example, let $\boldsymbol{DS}_e = (DS_e, I_{H_e})$ be the falling body model, where H_e = the phenomenon of free fall, and let $H_p = (F_p, B_{F_p})$ be *the phenomenon of projectile motion*, where its functional description F_p and its application domain B_{F_p} are specified as follows. The abstract type of real system AS_{F_p} is a medium size body in the vicinity of the earth, and it is thus identical to AS_{F_e}. However, the causal interaction scheme CS_{F_p} is more general than CS_{F_e}, for it consists in the body's being released at an arbitrary instant, and with *any velocity* and position (relative to the earth surface) whose respective values are within appropriate boundaries. B_{F_p} is then the set of all concrete medium size bodies in the vicinity of the earth that satisfy the given more general scheme of causal interactions.

Let us then consider the following system of four ordinary differential equations $\langle dx(t)/dt = \dot{x}(t),\ dy(t)/dt = \dot{y}(t),\ d\dot{x}(t)/dt = 0,\ d\dot{y}(t)/dt = -\boldsymbol{g}\rangle$, where $\boldsymbol{g}$ is a fixed real positive constant. The solutions of such equations uniquely determine the dynamical system $DS_p = (X \times Y \times \dot{X} \times \dot{Y}, (g^t)_{t \in T})$, where $X = Y = \dot{X} = \dot{Y} = T = R$ (the real numbers) and, for any t, x, y, $\dot{x}$, $\dot{y} \in R$, $g^t(x, y, \dot{x}, \dot{y}) = (\dot{x}t + x, -\boldsymbol{g}t^2/2 + \dot{y}t + y, \dot{x}, -\boldsymbol{g}t + \dot{y})$.

Let I_{H_p} be the following interpretation of DS_p on the phenomenon of projectile motion H_p. In the first place, for any projectile a, let p_a be the point where a is initially released; we then consider the plane that contains a's initial velocity vector and the earth center. On this plane, we fix both the x-axis and the y-axis of a Cartesian coordinate system, in such a way that its origin

[10] Quite obviously, if $\boldsymbol{g}$ = *standard gravity* ($\boldsymbol{g}$ = 9.80665 m/s^2), the model $(DS_e, I_{H_e}) = \boldsymbol{DS}_e$ turns out to be empirically correct within limits of precision sufficient for many practical purposes.

coincides with the earth center, and the y-axis passes through p_a. The positive direction of the y-axis is from the earth center to its surface; accordingly, we call the y-axis the *vertical axis*, and the x-axis the *horizontal axis*. We then interpret the first component X of the state space of DS_p as the set of all possible values of the *horizontal position* of the projectile a, the second component Y as the set of all possible values of its *vertical position*, the third component $\dot{X}$ as the set of all possible values of its *horizontal velocity*, the fourth component $\dot{Y}$ as the set of all possible values of its *vertical velocity*, and the time set T of DS_p as the set all possible values of *physical time*. Since all five of these magnitudes are measurable or detectable properties of the intended phenomenon H_p, I_{H_p} is an *empirical* interpretation of DS_p on H_p.

Let $\boldsymbol{DS}_p = (DS_p, I_{H_p})$; $\boldsymbol{DS}_p$ will be called the *projectile model*. By the respective definitions of B_{F_e} and B_{F_p}, $B_{F_e} \subset B_{F_p}$. Thus, by case 1, to show that the falling body model $\boldsymbol{DS}_e$ is reduced to the projectile model $\boldsymbol{DS}_p$, it suffice to exhibit an emulation u of DS_e in DS_p. Let u: $Y \times \dot{Y} \to X \times Y \times \dot{X} \times \dot{Y}$ and, for any y, $\dot{y} \in R$, $u(y, \dot{y}) = (0, y, 0, \dot{y})$; then, quite obviously, u is an emulation of DS_e in DS_p.

Case 2. Let us suppose next that $B_{F_2} \cap B_{F_1} = \emptyset$. In this case, no matter how DS_1 and DS_2 are related, $\boldsymbol{DS}_2$ *is not reduced to* $\boldsymbol{DS}_1$. For, even if DS_2 is identical to DS_1, *any* concrete system described by $\boldsymbol{DS}_2$ (that is to say, any concrete system $RS \in B_{F_2}$) is not a system also described by $\boldsymbol{DS}_1$.

Case 3. The case $B_{F_2} \cap B_{F_1} \neq \emptyset$ and $\neg(B_{F_2} \subseteq B_{F_1})$ is still left. This case is a combination of the previous two. In fact, for some concrete system $RS \in B_{F_2}$, both the $\boldsymbol{DS}_2$ and the $\boldsymbol{DS}_1$ descriptions apply to RS; however, if $RS \in B_{F_2}$ and $RS \notin B_{F_1}$, only the $\boldsymbol{DS}_2$ description applies to RS. Thus, in this case, if DS_1 emulates DS_2, $\boldsymbol{DS}_2$ *is incompletely reduced to* $\boldsymbol{DS}_1$.

We have just seen that case 3 only grants *incomplete* reduction of $\boldsymbol{DS}_2$ to $\boldsymbol{DS}_1$, provided that DS_1 emulates DS_2. However, $\boldsymbol{DS}_2$ may turn out to be *multiply* reduced to a *family* $(\boldsymbol{DS}_j)_{j \in J} = ((DS_j, I_{H_j}))_{j \in J}$ of empirically interpreted dynamical systems, each of which satisfies case 3 and emulates DS_2. This will be the case if the application domain B_{F_2} is included in the union of all application domains B_{F_j}. More precisely, for $\boldsymbol{DS}_2$ to *be multiply reduced to* $(\boldsymbol{DS}_j)_{j \in J}$, it is sufficient that, for any $j \in J$, $B_{F_2} \cap B_{F_j} \neq \emptyset$, $\neg(B_{F_2} \subseteq B_{F_j})$, DS_j emulates DS_2, and $B_{F_2} \subseteq \cup_{j \in J} B_{F_j}$.

A relationship between this condition for multiple reduction and the second order property version of the multiple realization concept [17, pp. 19–20, 103–104] is worth noticing. According to the latter, a property P *is multiply realized by properties of type* D just in case, for any x, x has P iff there is a property P_j of type D such that x has P_j. Any property P_j that satisfies the previous condition is said a *D-realizer* of the property P, and the property P itself is said a *second order property*.

Suppose now that $\boldsymbol{DS}_2$ is multiply reduced to $(\boldsymbol{DS}_j)_{j \in J}$ according to the previously stated sufficient condition. Let P_2 be the property that corresponds to functional description F_2 and, for any $j \in J$, P_j be the property that corresponds to functional description F_j. Let D be the property of being one of the properties P_j, for some $j \in J$. As $B_{F_2} \subseteq \cup_{j \in J} B_{F_j}$, it follows that, if x has P_2, then x has P_j, for some $j \in J$. Furthermore, if $B_{F_2} = \cup_{j \in J} B_{F_j}$, the converse holds as well, so that P_2 is multiply realized by properties of type D, and $(P_j)_{j\in J}$ is the family of its D-realizers.

From an intuitive point of view, the emulation relationship holds between two dynamical systems DS_1 and DS_2 when the *whole* dynamics of DS_2 is *exactly* reproduced by DS_1. I have argued so far that this relationship might be the basis for a new approach to reduction, which I have called *representational*. However, it is well known that, in many cases of inter-theoretic reduction, the relationship between the reduced theory S_2 and the reducing one S_1 is such that S_2 is only *partially* and *approximately* reduced to S_1. Furthermore, such a relationship typically is an *asymptotic* one, that is, it depends on some parameter p^* of either S_1 or S_2 in such a way that, for p^* tending to some fixed limiting value p, S_2 tends to be partially and approximately reduced to S_1, as established according to the limiting value p.[11]

The simple form of the emulation relationship considered so far may very well be the basis for a representational account of *total* and *exact* reduction (like, for example, the reduction of the falling body model $\boldsymbol{DS}_e$ to the projectile model $\boldsymbol{DS}_p$; see case 1 above). Nevertheless, we need a more sophisticated version of emulation for dealing with cases of asymptotic, partial and approximate reduction. In the next section, I suggest how this might be accomplished and provide (i) a formal definition of *partial* and *approximate* emulation, (ii) a simple example that shows how this relationship may turn out to be asymptotic, and (iii) sufficient conditions for partial and approximate reduction in empirically interpreted dynamical systems.

5 Partial and approximate emulation—sufficient conditions for partial and approximate reduction in empirically interpreted dynamical systems

Intuitively, a dynamical system $DS_1 = (M, (g^t)_{t\in T})$ partially emulates a second dynamical system $DS_2 = (N, (h^v)_{v\in V})$ if DS_1 exactly reproduces the dynamics of DS_2, limited to a fixed non-empty subset C of DS_2's state space N. This concept is thus a straightforward relativization of def. [8]. Let $C \neq \emptyset$, $C \subseteq N$, and define:

[9] DS_1 *C-emulates* DS_2 iff there is an injective function u: $C \to M$ such that for any $v \in V$, for any $c \in C$, there is $t \in T$ such that $u(h^v(c)) = g^t(u(c))$. Any function u that satisfies the previous condition is called a *C-emulation of* DS_2 *in* DS_1, and C is called its *emulation domain*.

Intuitively, DS_1 approximately emulates DS_2 if each state transition h^v: $y \to y$' of DS_2 approximately corresponds to a state transition g^t: $x \to x$' of

[11] Hooker ([15, p. 436]) maintains that "asymptotics provides the ground on which claims about inter-theoretic explanation, reduction and emergence must ultimately rest". According to him, "in physics, we find that the most famous theory pairs are all asymptotically related" ([15, p. 437]. Among such pairs, he explicitly mentions: (i) special relativity and Newtonian mechanics; (ii) optics and ray optics; (iii) quantum mechanics and Newtonian mechanics; (iv) statistical mechanics and thermodynamics (where, in each pair, the first element is the *reducing* theory and the second element is the *reduced* theory). According to Hooker, an analogous relationship may also hold between two different models of the *same* theory; an example is the following pair of models of Newtonian mechanics: a non linear classic pendulum model and a harmonic oscillator model ([15, p. 438]).

DS_1. This idea can be made precise by requiring that, for some injective function u, $u(y) = x$, and $u(y')$ be *sufficiently close* to x', where the two states $u(y')$, $x' \in M$ are sufficiently close to each other if their *distance* does not exceed a fixed non-negative real δ. Thus, the concept of approximate emulation in fact presupposes that M (i.e., the state space of DS_1) be equipped with a metric. Let d: $M \times M \rightarrow R^+$ be a metric on M, let $\delta \in R^+$. We then define:

[10] DS_1 *δ-emulates* DS_2 iff there is an injective function u: $N \rightarrow M$ such that, for any $v \in V$, for any $c \in N$, there is $t \in T$ such that $d(u(h^v(c)), g^t(u(c))) \leq \delta$. Any function u that satisfies the previous condition is called a *δ-emulation of DS_2 in DS_1*, and δ is called its *approximation degree*. If, for some δ, u is a δ-emulation of DS_2 in DS_1, the minimum of all such δ must exist, for R satisfies the least upper bound property.[12] Let δ^{min} be such a minimum; δ^{min} is then called *u's best approximation degree*. Thus, obviously, if, for some δ, u is a δ-emulation of DS_2 in DS_1, then u is a δ^{min}-emulation of DS_2 in DS_1.

Finally, by combining definitions [9] and [10], we get a definition of the intuitive idea of *partial* and *approximate* emulation. Let $C \neq \emptyset$, $C \subseteq N$, d: $M \times M \rightarrow R^+$ be a metric on M, and $\delta \in R^+$;

[11] DS_1 *C-δ-emulates* DS_2 iff there is an injective function u: $C \rightarrow M$ such that for any $v \in V$, for any $c \in C$, there is $t \in T$ such that $d(u(h^v(c)), g^t(u(c))) \leq \delta$. Any function u that satisfies the previous condition is called a *C-δ-emulation of DS_2 in DS_1*, C is called its *emulation domain*, and δ is called its *approximation degree*. If, for some δ, u is a C-δ-emulation of DS_2 in DS_1, the minimum of all such δ, indicated by δ^{min}, is called *u's best approximation degree*.[13] Thus, obviously, if, for some δ, u is a C-δ-emulation of DS_2 in DS_1, then u is a C-δ^{min}-emulation of DS_2 in DS_1.

5.1 An example

Let $X = \dot{X} = T = V = R$ (the real numbers), and $DS_n = (X \times \dot{X}, (g^t)_{t \in T})$ be the dynamical system that is determined by the solutions of the following non-linear system of ordinary differential equations $\langle dx(t)/dt = \dot{x}(t), d\dot{x}(t)/dt = -\boldsymbol{g} sin(x(t)/l) \rangle$, where $\boldsymbol{g}$ is a fixed real positive constant, and l is an arbitrary real positive parameter; note that this system is in fact a *non-linear classic pendulum*.[14] On the other hand, let $DS_o = (X \times \dot{X}, (h^v)_{v \in V})$ be the dynamical system that is determined by the solutions of the following linear system of ordinary differential equations $\langle dx(v)/dv = \dot{x}(v), d\dot{x}(v)/dv = -\boldsymbol{g} x(v)/l \rangle$, where $\boldsymbol{g}$ and l are as above; this second system is a *harmonic oscillator*.

Let $0 \leq \theta \leq \pi$, and $C_\theta = \{c$ such that, for some $x \in R$, $c = (x, 0)$, and $-\theta \leq x/l \leq \theta\}$. As it can be visually verified by means of any dynamical systems software, for an appropriately chosen $\delta_\theta > 0$, for any $v \in V$, for any $c \in C_\theta$, $d(h^v(c), g^v(c)) \leq \delta_\theta$, where d is the usual Euclidean distance on $X \times \dot{X}$

[12] According to the least upper bound property, for any non-empty subset A of R, if A has an upper bound, then the minimum of all upper bounds of A exists. Also recall that, for any $B \subseteq R$, *m is the minimum of B* iff $m \in B$ and, for any $b \in B$, $m \leq b$; *u is an upper bound of B* iff $u \in R$ and, for any $b \in B$, $b \leq u$.

[13] Such a minimum exists (see def. [10]).

[14] If $c = (m\pi l, 0)$ for some $m \in Z$, then $g^t(c) = c$ for any $t \in T$; that is to say, c is a fixed point of DS_n.

$= R^2$. Let u be the identity function on C_θ. By def. [11], it thus follows that u is a C_θ-δ_θ-emulation of DS_o in DS_n. Let δ_θ^{min} be the minimum of all such δ_θ. Then, by def. [11], u is a C_θ-δ_θ^{min}-emulation of DS_o in DS_n as well, and so DS_n C_θ-δ_θ^{min}-emulates DS_o.

It is important to keep in mind that δ_θ^{min} represents the best approximation degree to which DS_n partially emulates DS_o with respect to emulation domain C_θ. Besides, δ_θ^{min} is a function of $\theta \in [0, \pi]$. Therefore, we can study the behavior of δ_θ^{min} for θ tending to 0 from the right, and it is not difficult to verify that $\lim_{\theta \to 0^+} \delta_\theta^{min} = 0 = \delta_0^{min}$.

That is to say, for θ tending to 0 from the right, the best approximation degree to which DS_n partially emulates DS_o with respect to emulation domain C_θ tends to the best approximation degree to which DS_n partially emulates DS_o with respect to emulation domain C_0. In this precise sense, then, the relationship of partial and approximate emulation of DS_n by DS_o (with respect to emulation domain C_θ, and to the best approximation degree δ_θ^{min}) turns out to be asymptotic (see sec. 4, penultimate paragraph).

5.2 Empirical interpretations of the two dynamical systems of the previous example

Both dynamical systems $DS_n = (X \times \dot{X}, (g^t)_{t \in T})$ and $DS_o = (X \times \dot{X}, (h^v)_{v \in V})$ can be given natural empirical interpretations on corresponding phenomena. As regards the first system, let $H_n = (F_n, B_{F_n})$ be the phenomenon of the unrestricted swing of a pendulum or, more briefly, *the phenomenon of* (*unrestricted*) *pendulum swings*, where its functional description F_n and its application domain B_{F_n} are specified as follows . The abstract type of real system AS_{F_n} (called *simple* or *classic pendulum*) is made up of two structural elements, namely, a light rigid arm of length l, with a much heavier "bob" on one of its ends; the arm is pivoted on the other end, so that the bob can frictionlessly swing along a circular path of radius l in a vertical plane. The causal interaction scheme CS_{F_n} consists in releasing the bob at an arbitrary instant and position on its swinging path, with an arbitrary tangent velocity. B_{F_n} is then the set of all concrete simple pendula that satisfy the given scheme of causal interaction. Any such device will be called an (*unrestricted*) *pendulum*.

Let I_{H_n} be the following interpretation of DS_n on the phenomenon of unrestricted pendulum swings H_n. The first component X of the state space of DS_n is the set of all possible values of the *bob position*[15] of an arbitrary unrestricted pendulum, the second component $\dot{X}$ is the set of all possible values of the *bob tangent velocity*, and the time set T of DS_n is the set of all possible values of *physical time*. Since all three of these magnitudes are measurable or detectable properties of the intended phenomenon H_n, I_{H_n} is an empirical interpretation of DS_n on H_n, and $\boldsymbol{DS}_n = (DS_n, I_{H_n})$ is thus a model of H_n. This model is empirically correct, for an appropriate value of the constant

[15] A positive (negative) bob position x is the distance (the opposite of the distance), along the positive (negative) direction of the circular swinging path, of the bob itself from the intersection O between the path and the vertical straight line passing through the pendulum pivot. We take the positive path direction to be anticlockwise.

$\boldsymbol{g}$.[16] Henceforth, I will refer to $\boldsymbol{DS}_n$ as the (*unrestricted*) *pendulum model.*

As for the second system, let $H_{o\theta} = (F_{o\theta}, B_{F_{o\theta}})$ be the phenomenon of pendulum motion restricted to small swings or, more briefly, *the phenomenon of small pendulum swings*, where its functional description $F_{o\theta}$ and its application domain $B_{F_{o\theta}}$ are specified as follows. The abstract type of real system $AS_{F_{o\theta}}$ is a simple pendulum of length l, so it is identical to AS_{F_n}. However, the causal interaction scheme $CS_{F_{o\theta}}$ is more specific than CS_{F_n}, for it consists in releasing the pendulum's bob at an arbitrary instant, with zero tangent velocity, and in a position sufficiently close to the intersection O between the swinging path and the vertical straight line r passing through the pendulum pivot. This last clause can be put in the following form. Let θ $(0 \leq \theta \leq \pi)$ be the measure, in radians, of the angle between r and a straight line s passing through the pivot; let x be the bob's releasing position on the swinging path (where the origin is O, and the positive path direction is anticlockwise); then, $-\theta \leq x/l \leq \theta$. Thus, for θ sufficiently close to 0, the pendulum only performs small swings, when its bob is released at an arbitrary instant, with zero tangent velocity and in position x. $B_{F_{o\theta}}$ is then the set of all concrete simple pendula that satisfy the given more specific scheme of causal interaction. Any such device will be called a *small swing pendulum.*

Let $I_{H_{o\theta}}$ be the following interpretation of DS_o on the phenomenon of small pendulum swings $H_{o\theta}$. The first component X of the state space of DS_o is the set of all possible values of the *bob position* of an arbitrary small swing pendulum, the second component $\dot{X}$ is the set of all possible values of the *bob tangent velocity*, and the time set V of DS_o is the set all possible instants of *physical time.* These three magnitudes are measurable properties of the intended phenomenon $H_{o\theta}$. Therefore, $I_{H_{o\theta}}$ is an empirical interpretation of DS_o on $H_{o\theta}$, and $\boldsymbol{DS}_{o\theta} = (DS_o, I_{H_{o\theta}})$ is a model of $H_{o\theta}$. Furthermore, if θ is sufficiently small, such a model turns out to be empirically correct (for an appropriate value of the constant $\boldsymbol{g}$, see note 16). In what follows, $\boldsymbol{DS}_{o\theta}$ will be called the *small swing pendulum model.*

5.3 Sufficient conditions for partial and approximate reduction

Let us notice now that the unrestricted pendulum model $\boldsymbol{DS}_n$ and the small swing pendulum model $\boldsymbol{DS}_{o\theta}$ satisfy case 1 above (sec. 4), for $B_{F_{o\theta}} \subset B_{F_n}$ (by the definitions of the respective application domains B_{F_n} and $B_{F_{o\theta}}$). Moreover, we have seen (sec. 5.1, par. 2) that, for any $\theta \in [0, \pi]$, DS_n C_θ-δ_θ^{min}-emulates DS_o. The question then naturally arises whether this condition is sufficient for reduction of $\boldsymbol{DS}_{o\theta}$ to $\boldsymbol{DS}_n$.

Let us notice first that C_θ is, on the one hand, the emulation domain with respect to which dynamical system DS_n partially emulates DS_o and, on the other hand, C_θ is determined by the specific causal interaction scheme $CS_{F_{o\theta}}$ of the phenomenon of small pendulum swings. As a consequence, C_θ can be thought as singling out that part of the structure of DS_o that has an empirical interpretation according to $I_{H_{o\theta}}$. Let us call $E(C_\theta) = \{e\colon\ e = (c, h^v(c))$, for some $c \in C_\theta$ and some $v \in V\}$ *the empirical substructure of* DS_o *relative to*

[16] $\boldsymbol{g} = 9.80665$ m/s^2 (standard gravity) will be appropriate for many purposes.

interpretation $I_{H_{o\theta}}$.[17] Thus, by def. [11], the *whole* empirical substructure $E(C_\theta)$ is represented, through a partial emulation function u,[18] by corresponding structure of DS_n, within approximation degree δ_θ^{min}. Suppose now that $\Delta > 0$ is the desired approximation degree. Then, if $\delta_\theta^{min} \leq \Delta$, we can safely conclude that $\boldsymbol{DS}_{o\theta}$ is reduced to $\boldsymbol{DS}_n$.

In this connection, also recall that $\lim_{\theta \to 0^+} \delta_\theta^{min} = 0$, where $\theta \in [0, \pi]$ (sec. 5.1, par. 3). This means that the best approximation degree δ_θ^{min} to which the empirical substructure $E(C_\theta)$ is represented by corresponding structure of DS_n can be made as small as we please, by taking a sufficiently small value of θ. More precisely, for any desired approximation degree $\Delta > 0$, there is a sufficiently small θ_Δ such that, for any θ, if $0 < \theta < \theta_\Delta$, then $\delta_\theta^{min} < \Delta$. In addition, recall that $\delta_0^{min} = 0$ (sec. 5.1, par. 3); therefore, for any $\theta < \theta_\Delta$, $\delta_\theta^{min} < \Delta$. It thus follows that, for any $\theta < \theta_\Delta$, $\boldsymbol{DS}_{o\theta}$ is reduced to $\boldsymbol{DS}_n$.

In the general case, let $H = (F, B_F)$ be an arbitrary phenomenon, and $\boldsymbol{DS} = (DS, I_H)$ be any empirically interpreted dynamical system such that $\boldsymbol{DS}$ is an empirical model of H; let $DS = (M, (g^t)_{t \in T})$. Let us assume that, in force of interpretation I_H, there is a one-to-one correspondence between the *initial conditions* specified by the causal interaction scheme CS_F of phenomenon H (see sec. 4, par. 2) and a set of states of the dynamical system DS; let $C_F \subseteq M$ be such a set. Then, C_F is called *the empirical domain of* DS *relative to interpretation* I_H, and $E(C_F) = \{e : e = (c, g^t(c))$, for some $c \in C_F$ and some $t \in T\}$ is called *the empirical substructure of* DS *relative to* I_H.

Let $H_1 = (F_1, B_{F_1})$ and $H_2 = (F_2, B_{F_2})$ be two phenomena, and $\boldsymbol{DS}_1 = (DS_1, I_{H_1})$ and $\boldsymbol{DS}_2 = (DS_2, I_{H_2})$ be two empirically interpreted dynamical systems such that $\boldsymbol{DS}_1$ is an empirical model of H_1 and $\boldsymbol{DS}_2$ is an empirical model of H_2. Let C_{F_2} be the empirical domain of DS_2 relative to I_{H_2}, and $\Delta > 0$ be the desired approximation degree for DS_1 C_{F_2}-δ-emulating DS_2. The previous example thus suggests that case 1 (sec. 4) be supplemented with a weaker sufficient condition for reduction, as follows.

Case 1a. Let us suppose that $B_{F_2} \subseteq B_{F_1}$. If DS_1 C_{F_2}-δ-emulates DS_2 and $\delta \leq \Delta$, then $\boldsymbol{DS}_2$ is reduced to $\boldsymbol{DS}_1$.

A corresponding weaker condition can also be given for the case of *incomplete* reduction (case 3, sec. 4), as follows.

Case 3a. Suppose that $B_{F_2} \cap B_{F_1} \neq \emptyset$ and $\neg(B_{F_2} \subseteq B_{F_1})$. If DS_1 C_{F_2}-δ-emulates DS_2 and $\delta \leq \Delta$, then $\boldsymbol{DS}_2$ *is incompletely reduced to* $\boldsymbol{DS}_1$.

As for multiple reduction to a family $(\boldsymbol{DS}_j)_{j \in J} = ((DS_j, I_{H_j}))_{j \in J}$ of empirically interpreted dynamical systems, we get the following weaker condition. For $\boldsymbol{DS}_2$ to *be multiply reduced to* $(\boldsymbol{DS}_j)_{j \in J}$, it is sufficient that, for any $j \in J$, $B_{F_2} \cap B_{F_j} \neq \emptyset$, $\neg(B_{F_2} \subseteq B_{F_j})$, DS_j C_{F_2}-δ-emulates DS_2, $\delta_j \leq \Delta$, and $B_{F_2} \subseteq \cup_{j \in J} B_{F_j}$.

[17] See van Fraassen 1980 for a general discussion of the concept of an empirical substructure.

[18] Recall that, in this particular case, u is the identity function on $C_{\phi,\theta}$.

6 Concluding remarks

I have argued in this paper that reduction is better analyzed in terms of a *representational* relationship between *models*, rather than a *deductive* relationship between *theories*. Contrary to the received view, reduction has been conceived as a manifestation of an underlying representational relationship between mathematical models, namely, the one of *emulation*.

The representational theory of reduction has been developed so far only for the special case of dynamical systems (either empirically interpreted, or not). But, even in this special form, the theory is far from being complete.

Furthermore, even a complete representational theory for dynamical systems would not be sufficient to account for all relevant cases of reduction, for many models in real science are not of this kind. What we need is a *general* representational theory, as precise as the one restricted to dynamical systems, which apply to *arbitrary models*. The formulation of such a general theory, however, is not an easy matter, for it involves a preliminary investigation of fairly hard questions like: What is, *in general*, a purely mathematical model?[19] What is a structure preserving mapping between two *arbitrary* mathematical models? What is the relationship between two *arbitrary* mathematical models that generalizes the one of emulation between dynamical systems? What is, *in general*, an empirical interpretation of a mathematical model on a phenomenon?

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[19] In sec. 3 (see B1), I defined a mathematical model *MS* as a set D together with a family $(\sigma_i)_{i \in I}$ of relations on D. This definition is fine as far as *relational* models are concerned, but not all mathematical models are of this kind. For instance, a topological space (with the standard axiomatization in terms of open sets) is not a relational model. Bourbaki 1968 (ch. 4) contains a quite general treatment of mathematical structures. However, Bourbaki's general theory of structures is developed at the metamathematical level. What we need is a theory of models developed *within* set theory, and thus at the mathematical level, as general as Bourbaki's metamathematical theory of structures.

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