Dynamical Systems on Monoids: Toward a General Theory of Deterministic Systems and Motion

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Summary

GOAL

(G) Generalize the standard notion of a dynamical system

THESIS

A monoid structure on time is both

1. necessary for (G) and
2. sufficient for (G)

The standard definition of a dynamical system

Let Z = integers, Z^+ = integers ≥ 0, R = reals, R^+ = reals ≥ 0.

Definition 0: *DS is a dynamical system* iff *DS* is a pair $(M, (g^t)_{t\in T})$ such that

(i) T is either Z, Z^+, R, or R^+. Any $t \in T$ is called a *duration* of the system, and T is called its *time set*;

(ii) M is a non-empty set. Any $x \in M$ is called a *state* of the system, and M is called its *state space*;

(iii) $(g^t)_{t\in T}$ is a family indexed by T of functions from M to M. For any $t \in T$, the function g^t is called the *state transition of duration t* (briefly, *t-transition*, or *t-advance*) of the system;

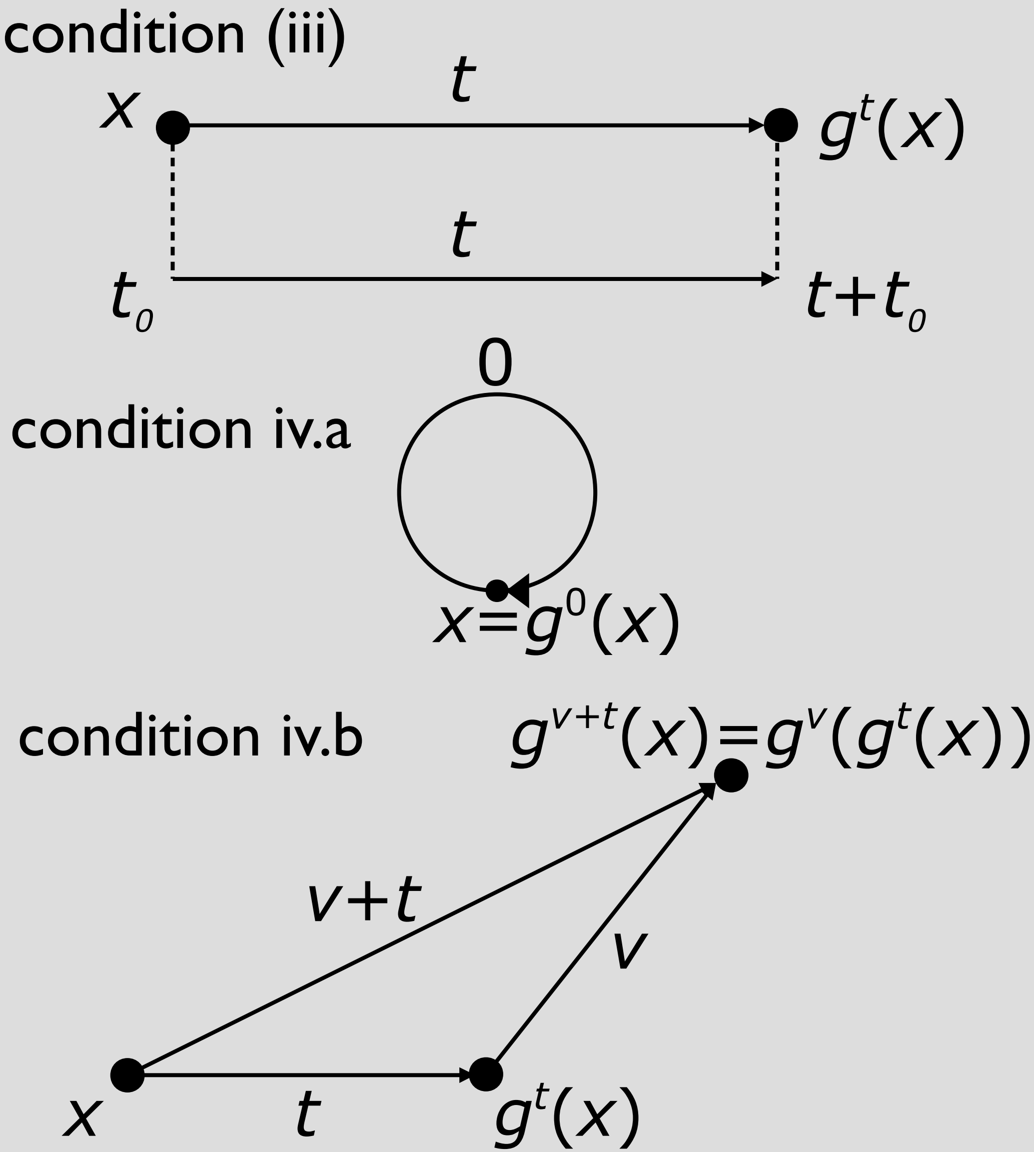
(iv) for any $v, t \in T$, for any $x \in M$,

$g^0(x) = x$;

$g^{v+t}(x) = g^v(g^t(x))$.

Definition 0 captures the intuitive notion of a **deterministic system**

- (iii): gives us the state of the system after an evolution of an arbitrary duration $t \in T$, if the state x at the present instant $t_0 \in T$ is known
- (iv.a): whatever state x the system is in, the evolution of duration 0 does not modify that state
- (iv.b): any evolution of duration $v+t$ can be decomposed in two successive evolutions, the first one of duration t, and the second one of duration v



Examples of systems that satisfy definition 0

- Discrete time and discrete state space
  - Turing machines
  - Cellular automata (with a finite number of non-quiescent cells)
- Discrete time and continuous state space
  - Many systems specified by difference equations
  - Iterated mappings on *R* (*R* = real numbers)
- Continuous time and continuous state space
  - Systems specified by ordinary differential equations
  - Many neural networks

Why definition 0 is not enough

- Definition 0 is not fully explicit, for it does not make clear *exactly which structure* on the time set T is needed, in order to support appropriate dynamics for the system.
- By condition (i), T is either Z, Z^+, R, or R^+.
- With respect to the addition operation, these four models share the structure of a linearly ordered commutative monoid;
- but it is by no means obvious that *all* this structure on T is needed for a *general* definition of a dynamical system.
- We maintain that the minimal structure on the time set that underpins a materially adequate definition of a dynamical system is just that of a *monoid* (see Definition 1, next slide).

The definition of a dynamical system on a monoid

Definition 1: *DS_L is a dynamical system on L* iff DS_L is a pair $(M, (g^t)_{t\in T})$ and L is a pair $(T, +)$ such that

(i) $L = (T, +)$ is a monoid. Any $t \in T$ is called a *duration* of the system, T is called its *time set*, and L its *time model*;

(ii) M is a non-empty set. Any $x \in M$ is called a *state* of the system, and M is called its *state space*;

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A monoid is necessary for an adequate definition of a DS_L

- **THESIS 1:** the monoid structure on the time set T cannot be further weakened.
- To support this claim, we will consider first
  - the directed graph that any dynamical system induces on its state space, and
  - a revealing link between this graph and category theory.
- We will then discuss the following result:
  - Such a graph can be made into a category if, and only if, the algebraic structure on the time set T is that of a monoid (Theorem 1, dia 12).

The **transition graph** of a DS_L

- The *transition graph* of a DS_L is a directed and labeled graph that depicts the whole dynamics of the system.
- Each *point* of the graph corresponds to exactly one state of the DS_L, while each *arrow* stands for a state transition from its *source* to its *target*.
- Each arrow is *labeled* with the *duration* of the corresponding state transition.

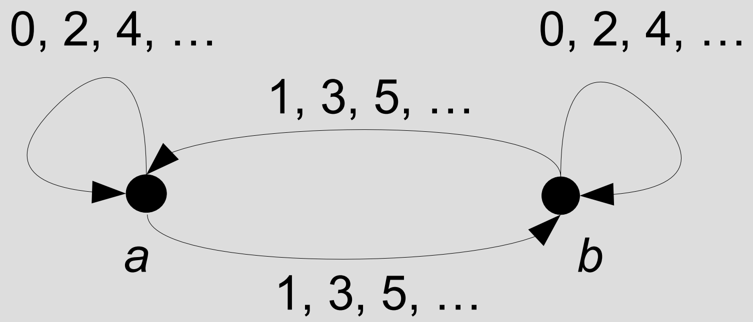
Example 0:

$DS_L = (\{a, b\}, (g^t)_{t \in Z^+})$

$L = (Z^+, +)$

if t is even, $g^t(x) = x$

if t is odd, $g^t(x) = y$, $y \neq x$



Transition graphs and a link to category theory

- The transition graph can be rigorously defined for *any* dynamical system DS_L on a monoid $L = (T, +)$.
- But it can also be defined for any *quasi*-DS_L – a system that differs from a dynamical system just for the fact that the binary operation + not necessarily is associative; *i.e.*, the time model L is not assumed to be a monoid, but a *magma with unity*.
- This possibility opens up an interesting link to category theory because, for any such graph,
  - any two consecutive arrows can be composed;
  - a family of identity arrows always exists.

The transition graph of a DS_L is a category – nec. and suf. conds

- **Theorem 1:** the transition graph of a quasi-DS_L is a category iff L is a monoid (or, equivalently, DS_L is a dynamical system on L).
- Theorem 1 provides us with a justification for our claim that the *minimal* structure on the time set that supports a materially adequate definition of a dynamical system is *at least* that of a monoid.
- For, if it is not, the transition graph of the system cannot even be made into a category.

A monoid is sufficient for an adequate definition of a DS_L

THESIS 2: the monoid structure on the time set *T is* sufficient to support a variety of significant dynamical concepts, as well as a rich web of relations among them.

- To substantiate this claim, we
  - define a considerable number of genuine dynamical concepts;
  - go through a series of meaningful and sometimes even surprising results about them.

x-motion of a DS_L

Let Y and Z be any two sets; let *eval*: $Z^Y \times Y \to Z$ such that, for any function $f \in Z^Y$, for any $y \in Y$, $eval(f, y) = f(y)$; we then define:

Definition 9:
g_x *is the motion with initial state x of* DS_L iff $DS_L = (M, (g^t)_{t \in T})$ is a dynamical system on $L = (T, +)$, for any $x \in M$, for any $t \in T$, $g_x: T \to M$ and $g_x(t) = eval(g^t, x)$. (g_x is also called *the x-motion of* DS_L or *the x-evolution of* DS_L.)

For any $x \in M$, g_x represents the motion of the system when the state at the initial instant t_0 is x – *i.e.*, if the state at t_0 is x, then, for any $t \in T$, $g_x(t)$ is the state at instant $t + t_0$.

Orbits and phase portrait

Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$, and $x \in M$.

Definition 10:

(i) *The orbit of* $x = orb(x) = \{z: z = g^t(x)$, for some $t \in T\}$;

(ii) *r is an orbit* iff for some $x \in M$, $r = orb(x)$;

(iii) *the phase portrait* $= \{r: r$ is an orbit$\}$.

No crossing orbits

From an intuitive point of view, if a system is deterministic and two orbits have a state in common, then, from that state on, they must coincide; that is to say, deterministic systems do not admit crossing orbits. The following proposition expresses exactly this fact.

Proposition 4: Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on $L = (T, +)$. For any $x, y, z \in M$, if $z \in orb(x)$ and $z \in orb(y)$, then $orb(z) \subseteq orb(x)$ and $orb(z) \subseteq orb(y)$.

Consequences of the existence of the inverse −*t* of a duration *t*

As the time model $L = (T, +)$ of a dynamical system DS_L is a monoid, for any $t \in T$, there is at most one inverse; when the inverse of t exists, we indicate it by $-t$. The next proposition highlights three interesting consequences of the existence of the inverse $-t$ of a duration $t \in T$.

Proposition 5: Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on $L = (T, +)$. For any $t \in T$, if the inverse $-t$ of t exists, then
(i) g^t is a bijection;
(ii) g^{-t} is the inverse of g^t with respect to the operation of function composition;
(iii) $g^{-t} = (g^t)^{-1}$, where $(g^t)^{-1}$ is the inverse function of g^t.

Further definitions – orbit types

We just mention that it is also possible to

- define the concepts of fixed and periodic point, as well as the period of a periodic point;
- distinguish six mutually exclusive and jointy exhaustive orbit types:

  (1) periodic and merging;
  (2) periodic and not merging;
  (3) eventually periodic and merging;
  (4) eventually periodic and not merging;
  (5) aperiodic and merging;
  (6) aperiodic and not merging.

It is then interesting to study how the instantiation pattern of the six orbit types varies with specific characters of the systems considered.

Past and future

Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$; we introduce purely dynamical concepts of *past* and *future*, as follows. Let 0 be the unity of L, and $x \in M$.

Definition 14:

(i) $P^t(x)$ *is the t-past of x* iff $t \in T-\{0\}$ and $P^t(x) = \{y: g^t(y) = x\}$;

(ii) $F^t(x)$ *is the t-future of x* iff $t\in T-\{0\}$ and $F^t(x) = \{y: g^t(x) = y\}$;

(iii) $P(x)$ *is the past of x* iff $P(x) = \cup_{t\in T-\{0\}} P^t(x)$;

(iv) $F(x)$ *is the future of x* iff $F(x) = \cup_{t\in T-\{0\}} F^t(x)$.

Note that, by Definition 14.ii, for any $x \in M$ and $t \in T-\{0\}$, $F^t(x)$ is a singleton. Analogous definitions can be given for a set of states $X \subseteq M$.

Eight concepts of reversibility (1/2)

We can distinguish at least *eight* notions of reversibility. Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on $L = (T, +)$.

Definition 15: (part 1/2)

(R) DS_L *is reversible* iff for any $x \in M$, for any $t \in T$, for some $w \in T$, $g^w(g^t(x)) = x$;

(SR) DS_L *is strictly reversible* iff for any $t \in T$, for some $w \in T$, for any $x \in M$, $g^w(g^t(x)) = x$;

(LR) DS_L *is logically reversible* iff for any $t \in T$, g^t is injective;

(CP) DS_L *has complete past* iff for any $t \in T$, g^t is surjective;

(CLR) DS_L *is completely logically reversible* iff for any $t \in T$, g^t is bijective;

Eight concepts of reversibility (2/2)

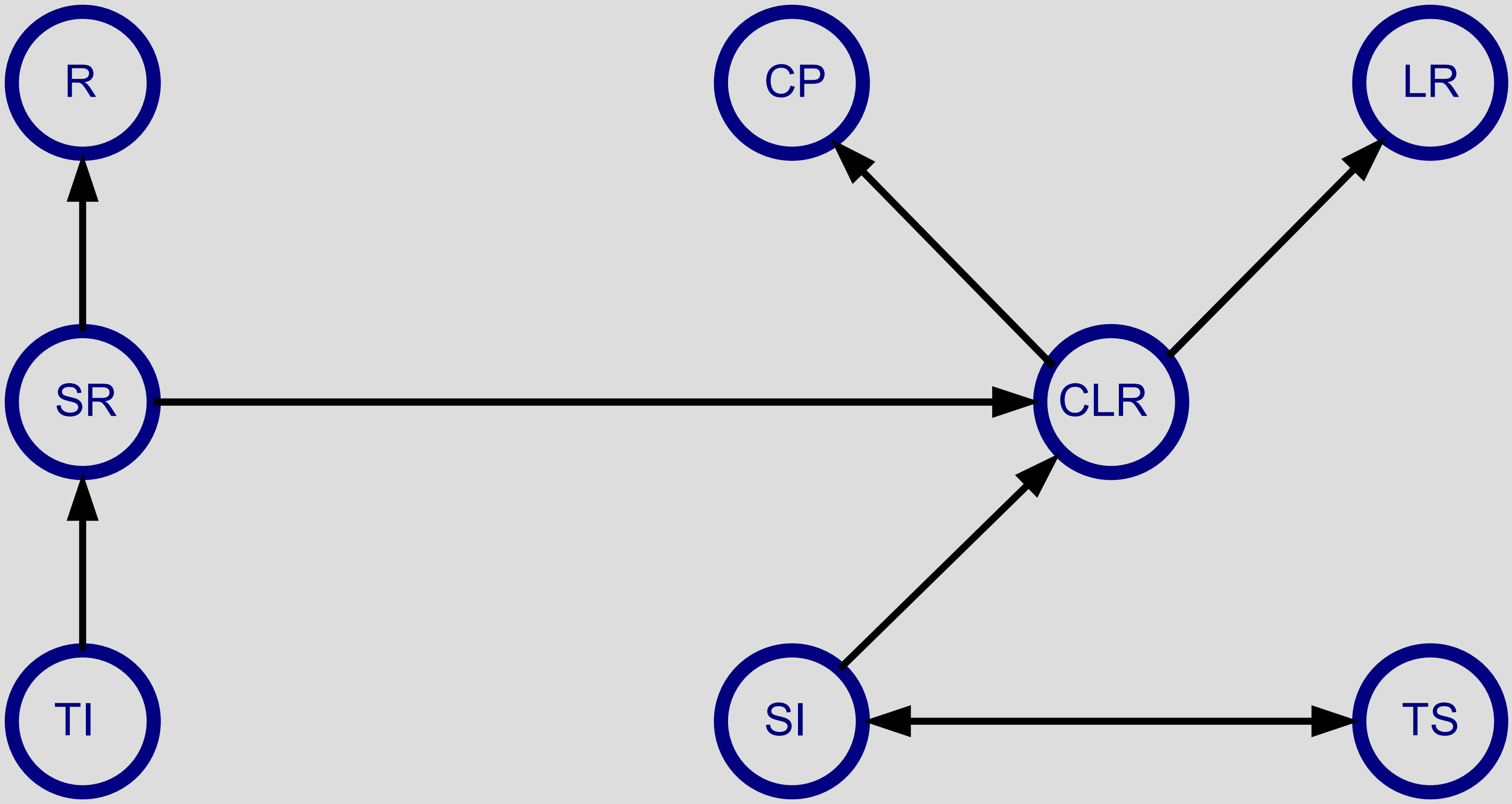
Definition 15: (part 2/2)

(TS) DS_L *is time symmetric* iff DS_L is completely logically reversible and there is $\sim: M \to M$ such that, for any $x \in M$, for any $t \in T$, $\sim(g^t(\sim(x))) = (g^t)^{-1}(x)$;

(SI) DS_L *is space invertible* iff there is $\sim: M \to M$ such that, for any $x \in M$, for any $t \in T$, $g^t(\sim(g^t(\sim(x)))) = x$;

(TI) DS_L *is time invertible* iff $L = (T, +)$ is a group.

Implications between the eight reversibility concepts



FINE

GRAZIE

THANKS