REVERSIBLE DYNAMICS AND THE DIRECTIONALITY OF TIME

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The received view on the problem of the direction of time holds it that time has no intrinsic dynamical properties, and that its apparent asymmetry, to be understood in purely topological terms, is dependent on the directional properties of physical processes. In this paper we shall challenge both claims, in the light of an algebraic representation of time. First, we will show how to give a precise formulation to the intuitive idea that time possesses an intrinsic dynamics; this formulation relies on the fact that the algebraic properties of time can equivalently be understood in dynamical terms. Second, we shall argue that the directional properties displayed by the processes occurring in time depend on the directional properties of time, rather than the converse.

Keywords: Arrow of time, Dynamical systems, Time systems, Reversibility.

1 From Algebra to Dynamics

According to a long-date tradition – committed, for the most part, with a causal theory of time and including authors such as Reichenbach (1956, 1958) [6,7], Grünbaum (1973) [3], Mehlberg (1980) [5] and van Fraassen (1970) [8] – philosophical analysis of physical time should primarily deal with its topological properties, such as continuity, homogeneity, anisotropy and linear ordering. While confronted with the problem of the arrow of time, exponents of this tradition typically come to the conclusion that (i) time has no dynamical direction – it does not "flow" – and (ii) its possessing a merely topological directionality, or anisotropy, ultimately rests on the factlike directionality of the physical processes occurring in it.

In this paper we shall challenge both claims, in the light of an algebraic representation of time. First, we will show how to give a precise formulation to the intuitive idea that time possesses an intrinsic dynamics; this formulation relies on the fact that the algebraic properties of time can equivalently be understood in dynamical terms. Second, we shall argue that the directional properties displayed by the processes occurring in time depend on the directional properties of time, rather than the converse.

1.1. *Dynamical Systems on Monoids*

An autonomous deterministic system is typically understood as a group of transformations on a given set, along with the postulate of their invariance with respect to time displacement (Lucas 1973 [4]; van Fraassen 1989 [9]). However, according to Giunti and Mazzola [2] (this volume), a dynamical system on a monoid is the minimal mathematical structure needed to capture the intuitive concept of a deterministic system. Giunti and Mazzola define a dynamical system on a monoid as follows:

Definition 1: (Dynamical system on a monoid)
DS_L is a dynamical system on L iff DS_L is a pair $(M, (g^t)_{t\in T})$ and $L = (T, +)$ is a monoid with identity 0 such that
(i) M is a non-empty set;
(ii) $(g^t)_{t\in T}$ is a family indexed by T of functions from M to M;
(iii) for any v, $t \in T$, for any $x \in M$,
  (a) $g^0(x) = x$;
  (b) $g^{v+t}(x) = g^v(g^t(x))$.

M represents the *state space* of DS_L; any $x \in M$ is a possible *state* the dynamical system can assume at a time. Each function g^t of the family $(g^t)_{t\in T}$ is a *state transition of duration t* (or *t-advance*) from states of DS_L to states of DS_L. Comparison between the dynamical properties of different dynamical systems is made possible by means of the following definitions.

Definition 2: (Isomorphism between dynamical dystems)
Let $DS_{L_1} = (M_1, (g^{t_1})_{t\in T_1})$ be a dynamical system on a monoid $L_1 = (T_1,+)$, and let $DS_{L_2} = (M_2, (g^{t_2})_{t\in T_2})$ be a dynamical system on a monoid $L_2 = (T_2,\oplus)$.
A function *f is a ρ-isomorphism of DS_{L_2} in DS_{L_1}* iff ρ: $T_2 \to T_1$ is a monoid isomorphism of L_2 in L_1 and f: $M_2 \to M_1$ is a bijection such that, for any $x_2 \in M_2$ and $t_2 \in T_2$, $f(g^{t_2}(x_2)) = g^{\rho(t_2)}(f(x_2))$.

Definition 3: (Isomorphic dynamical systems on monoids)
Let DS_{L_1} be a dynamical system on L_1 and DS_{L_2} be a dynamical system on L_2.
DS_{L_2} is isomorphic to DS_{L_1} iff there exist f and ρ such that f is a ρ-isomorphism of DS_{L_2} in DS_{L_1}.

Isomorphism is an equivalence relation on any given set of dynamical systems; notably, isomorphic dynamical systems describe the same dynamics. For this reason, isomorphic dynamical systems may well be understood as being identical.

1.2. *Time Models and Time Systems*

As Definition 1 makes clear, a monoid is indispensable in order to define a dynamical system, for it provides the suitable time model underlying its dynamics. This fact is not casual, for there exists a very tight relationship connecting the time evolution of deterministic systems and the algebraic properties of monoids.

In general, any monoid $L = (T, +)$ can be thought as a *time model* when the elements of its domain are conceived as durations, and its binary operation + as addition of any two durations. Furthermore, any monoid can be endowed with a dynamical representation of its own algebraic structure via the following definition.

Definition 4: (Time system of a monoid)
TS(*L*) *is the time system of L* iff $L = (T, +)$ is a monoid and $TS(L) = (I, (\iota^t)_{t\in T})$ is the ordered pair such that $I = T$ and, for any $t \in T$ and $i \in I$,
(i) $\iota^t: I \to I$;
(ii) $\iota^t(i) = t + i$.

Any $i \in I$ is called an *instant* of *TS*(*L*), while *I* itself is the set of all instants composing the time system *TS*(*L*). From an intuitive point of view, for any $t \in T$ and $i \in I$, the *arrow* $i \xrightarrow{t} \iota^t(i)$ (see sec. 3 of Giunti and Mazzola [2], this volume) is intended to represent the flowing of time from the present instant *i* to the instant $\iota^t(i)$ reached after *duration t*; therefore, on this interpretation, $\iota^t(i)$ cannot be anything else but the instant obtained by adding duration *t* to instant *i*, just as required by (ii) of Definition 4. It is easy to prove that the time system of any monoid is a dynamical system on the monoid itself.

Proposition 1: Let $TS(L) = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.
TS(*L*) is a dynamical system on *L*.

Proof: By Definition 4, *I* is a non-empty set and $(\iota^t)_{t\in T}$ is a family of functions on *I*, indexed by *T*; also, for any $i \in I$ and any $t, v \in T$,

$$\iota^0(i) = 0 + i = 0 \quad (1)$$

$$\iota^{t+v}(i) = (t + v) + i = t + (v + i) = \iota^t(\iota^v(i)) \quad (2)$$

Hence, by Definition 1, *TS*(*L*) is a dynamical system on *L*. *Q.E.D.*

As we mentioned before, isomorphic dynamical systems can be understood as describing the same dynamics. In the light of this, it is no surprise that time systems with isomorphic algebraic structures must themselves be isomorphic.

Proposition 2: Let $L_1 = (T_1, +)$ be a monoid with time system $TS(L_1)$, and $L_2 = (T_2, \oplus)$ be a monoid with time system $TS(L_2)$.
If ρ is a monoid isomorphism of L_2 in L_1, then ρ is a ρ-isomorphism of $TS(L_2)$ in $TS(L_1)$.

Proof: Let $\rho: T_2 \to T_1$ be a monoid isomorphism of L_2 in L_1. Hence, for any $t_2 \in T_2$ and any $i_2 \in I_2 = T_2$,

$$\rho(\iota^{t_2}(i_2)) = \rho(t_2 \oplus i_2) = \rho(t_2) + \rho(i_2) = \iota^{\rho(t_2)}(\rho(i_2)) \quad (3)$$

Hence, by Definition 2, ρ is a ρ-isomorphism of $TS(L_2)$ in $TS(L_1)$. *Q.E.D.*

Corollary 1: Let L_1 be a monoid with time system $TS(L_1)$, and L_2 be a monoid with time system $TS(L_2)$.
If L_2 is isomorphic to L_1, then $TS(L_2)$ is isomorphic to $TS(L_1)$.

Proof: If L_2 is isomorphic to L_1, there is a monoid isomorphism ρ of L_2 in L_1. By Proposition 2, ρ is a ρ-isomorphism of $TS(L_2)$ in $TS(L_1)$. Therefore, by Definition 3, $TS(L_2)$ is isomorphic to $TS(L_1)$. *Q.E.D.*

In general, any dynamical system DS_L can be endowed with an algebraic structure on the set of its state transitions. More precisely, this structure consists of such a set along with the standard operation of function composition. It is well known that, when L is a group (*e.g.* the real numbers), this algebraic structure is also a group, which is sometimes called the one parameter group of transformations of the dynamical system DS_L (Arnold 1998). However, in the general case, we refer to this structure as the *transition algebra* of DS_L. We will see below that, when L is just a monoid, the transition algebra of DS_L is a monoid as well.

Definition 5: (Transition algebra of a dynamical system)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.
$TA(DS_L)$ *is the transition algebra of* DS_L iff $TA(DS_L) = (H, \circ)$, where
$H = \{h : h = g^t$, for some $t \in T\}$ and $\circ$ is the standard operation of function composition.

The following proposition is a straightforward consequence of Definition 5.

Proposition 3: Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; let $TA(DS_L) = (H, \circ)$ be the transition algebra of DS_L.
$TA(DS_L)$ is a monoid with identity g^0.

Proof: For any $h, f \in H$, let $t, v \in T$ such that $h = g^t$, $f = g^v$; then

$$g^t \circ g^v = g^{t+v} \in H \tag{4}$$

For any $h, f, r \in H$, let $t, v, u \in T$ such that $h = g^t$, $f = g^v$, $r = g^u$; then

$$g^t \circ (g^v \circ g^w) = g^t \circ g^{v+w} = g^{t+(v+w)} = g^{(t+v)+w} = g^{t+v} \circ g^w = (g^t \circ g^v) \circ g^w \tag{5}$$

In the first place, $g^0 \in H$; in addition, for any $h \in H$, let $t \in T$ such that $h = g^t$; then

$$g^0 \circ g^t = g^{0+t} = g^t = g^{t+0} = g^t \circ g^0 \tag{6}$$

Hence, by (4) and (5), $TA(DS_L)$ satisfies closure with respect to the composition operation and associativity; by (6), g^0 is the identity element. Therefore, $TA(DS_L)$ is a monoid.
Q.E.D.

In general, the time model L of a dynamical system DS_L is epimorphic, but not isomorphic, to the transition algebra $TA(DS_L)$. However, in the special case of a time system $TS(L)$, L turns out to be isomorphic to $TA(TS(L))$.

Proposition 4: Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$ with identity 0; let $TA(DS_L) = (H, \circ)$ be the transition algebra of DS_L.
L is epimorphic to $TA(DS_L)$.

Proof: Let $\rho: T \to H$ be the family $(g^t)_{t \in T}$ itself. Then, for any t and $v \in T$,

$$\rho(t + v) = g^{t+v} = g^t \circ g^v \tag{7}$$

Also, ρ is obviously surjective; therefore, ρ is an epimorphism of L in $TA(DS_L)$, and thus L is epimorphic to $TA(DS_L)$. *Q.E.D.*

Corollary 2: Let $TS(L) = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0; let $TA(TS(L)) = (H, \circ)$ be the transition algebra of $TS(L)$.
L is isomorphic to $TA(TS(L))$.

Proof: Let $\rho: T \to H$ be the family $(g^t)_{t \in T}$ itself. By (7), ρ is a monoid homomorphism of L in $TA(TS(L))$. It thus suffice to prove that ρ is bijective.
For any $t, v \in T$, if $t \neq v$, then

$$t + 0 \neq v + 0$$
$$\iota^t(0) \neq \iota^v(0)$$
$$\iota^t \neq \iota^v$$
$$\rho(t) \neq \rho(v) \tag{8}$$

By Definition 5, for any $h \in H$, $h = \iota^t$ for some $t \in T$; but, by hypothesis,

$$\rho(t) = \iota^t \tag{9}$$

Thus, by (8), ρ is injective and, by (9), ρ is surjective. Therefore, L is isomorphic to $TA(TS(L))$. *Q.E.D.*

Let us also notice that, in general, a dynamical system DS_L is not isomorphic to the time system $TS(TA(DS_L))$ of its transition algebra, for isomorphism between dynamical systems requires their time models to be isomorphic, while Proposition 4 only grants epimorphism between L and $TA(DS_L)$. However, in the special case of time systems, $TS(L)$ does turn out to be isomorphic to $TS(TA(TS(L)))$, as shown by the following proposition.

Proposition 5: For any time system $TS(L)$, $TS(TA(TS(L)))$ is isomorphic to $TS(L)$.

Proof: Let $TS(L)$ be the time system of a monoid L, $TA(TS(L))$ be the transition algebra of $TS(L)$, and $TS(TA(TS(L)))$ be the time system of $TA(TS(L))$. By Corollary 2, L is isomorphic to $TA(TS(L))$. Hence, by Corollary 1, $TS(L)$ is isomorphic to $TS(TA(TS(L)))$. *Q.E.D.*

Proposition 1 and Corollary 2 jointly show that any monoid gives rise to a dynamical system of a special type (*i.e.* its time system) that preserves its algebraic properties. On the other hand, Proposition 3 and Proposition 5 show that any dynamical system of this special type (*i.e*. any time system) gives rise to an algebraic structure (*i.e*. its transition algebra) that preserves its dynamical properties. For this very reason, the study of the algebraic properties of a monoid L could also be carried out in the form of a study of the dynamical properties of its time system $TS(L)$, and vice versa.

2 Reversibility of Time Systems

There are at least six main different notions of *latu sensu* reversibility that may be applied to dynamical systems, namely: reversibility, logical reversibility, strict reversibility, complete past, complete logical reversibility, and time invertibility. In this section, we shall see how some of the basic algebraic properties of a monoid can determine the kind of reversibility displayed by the corresponding time system.

2.1. *Reversibility, Strict Reversibility, and Logical Reversibility*

The weakest form of indifference a dynamical system can display with respect to the direction of time is that of *reversibility*.

Definition 6: (Reversible dynamical system)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$.
DS_L *is reversible* iff for any $x \in M$, for any $t \in T$, for some $r \in T$, $g^r(g^t(x)) = x$.

Reversibility plays a crucial role in the present discussion, because of the following property.

Proposition 6: Let $TS(L) = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.
If $TS(L)$ is reversible, then any element of L has a left inverse.

Proof: Suppose $TS(L)$ is reversible. Then, by Definition 6, for any $i \in I$ and $t \in T$, there is $r \in T$ such that

$$\iota^r(\iota^t(i)) = r + t + i = i \tag{10}$$

Let $i = 0$; thus, by (17),

$$r + t = 0 \tag{11}$$

Hence, by (11), any $t \in T$ has a left inverse. *Q.E.D.*

The definition of a reversible dynamical system (Definition 6) only requires that, for the image $g^t(x)$ of state x by transition g^t, there is another state transition g^r leading back $g^t(x)$ to x. Under this condition, it may be the case that the images $g^t(x)$ and $g^t(y)$ of different states x and y need different state transitions in order to be led back to x and y, respectively. Requiring a reversible dynamical system to be *strictly reversible* is ruling out this possibility.

Definition 7: (Strictly reversible dynamical system)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$.
DS_L *is strictly reversible* iff for any $t \in T$, for some $r \in T$, for any $x \in M$, $g^r(g^t(x)) = x$.

Proposition 7: Let $TS(L) = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.
If any element of L has a left inverse, then $TS(L)$ is strictly reversible.

Proof: Suppose that, for any $t \in T$, there is $r \in T$ such that

$$r + t = 0 \tag{12}$$

Then, for any $j \in I$,

$$\iota^r(\iota^t(j)) = r + t + j = 0 + j = j \tag{13}$$

Hence, by (13) and Definition 7, $TS(L)$ is strictly reversible. *Q.E.D.*

Corollary 3: Let $TS(L) = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0. The following statements are equivalent.
(i) $TS(L)$ is reversible;
(ii) $TS(L)$ is strictly reversible
(iii) any element of L has a left inverse.

Proof: We show (i) → (iii) → (ii) → (i). Suppose $TS(L)$ is reversible; then, by Proposition 6, any element of L has a left inverse. Thus, by Proposition 7, $TS(L)$ is strictly reversible. Therefore, by Definition 7 and Definition 6, $TS(L)$ is reversible. *Q.E.D.*

Corollary 3 is a clear example of how the dynamical properties of time systems ultimately depend on the algebraic features of the corresponding monoids. Though generally not equivalent, in the special case of time systems, reversibility and strict reversibility coincide because of their equivalence to the same algebraic property – namely, that any element of a monoid possess a left inverse.

A necessary condition for a dynamical system to be strictly reversible is that all its state transitions be injective. For, if a state transition mapped different states x and z into a unique image y, then a unique state transition could not lead y back to both x and z. Strictly reversible dynamical systems must therefore display the following property.

Definition 8: (Logically reversible dynamical system)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$.
DS_L *is logically reversible* iff for any $t \in T$, g^t is injective.

Proposition 8: Let $TS(L) = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.
If $TS(L)$ is reversible, then $TS(L)$ is logically reversible.

Proof: Suppose $TS(L)$ is reversible; then, by Corollary 3, $TS(L)$ is strictly reversible. Furthermore, any strictly reversible dynamical system is logically reversible. Therefore, $TS(L)$ is logically reversible. *Q.E.D.*

The converse of Proposition 8, however, is false. A very simple counterexample is represented by the time system of the additive monoid $(Z^+, +)$, where Z^+ is the set of non-negative integers.

2.2. *Complete Past and Complete Logical Reversibility*

As we saw, logical reversibility requires all state transitions of a dynamical system to be injective. Its natural counterpart is the notion of *complete past*, according to which all state transitions are surjective.

Definition 9: (Dynamical system with complete past)
Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$.
DS_L *has complete past* iff for any $t \in T$, g^t is surjective.

However, in the case of time systems, logical reversibility and complete past do not play analogous roles. Rather, complete past must be understood as the correlate of reversibility and strict reversibility – as the following propositions make clear.

Proposition 9: Let $TS(L) = (I, (\iota^t)_{t\in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.
$TS(L)$ has complete past iff any element of L has a right inverse.

Proof: Suppose $TS(L)$ has complete past; then, by Definition 9, for any $t \in T$ and any $i \in I = T$ there exists $j \in I$ such that

$$\iota^t(j) = t + j = i \tag{14}$$

let $i = 0$; then, by (14),

$$t + j = 0 \tag{15}$$

Therefore, any element of L has a right inverse.
Conversely, suppose that, for any $t \in T$, there is $i \in I = T$ such that

$$t + i = 0 \tag{16}$$

then, for any $j \in I$,

$$\iota^t(i + j) = t + (i + j) = (t + i) + j = 0 + j = j \tag{17}$$

Hence, by Definition 9, $TS(L)$ has complete past. *Q.E.D.*

In the light of Corollary 3, Proposition 9 shows the role of complete past to be analogous to the one of reversibility and strict reversibility. This similarity can be carried even further, as time systems with complete past can ultimately be proved to be identical with reversible ones.

Proposition 10: Let $TS(L) = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.
$TS(L)$ is reversible iff $TS(L)$ has complete past.

Proof: Suppose $TS(L)$ is reversible; then, by Corollary 3, for any $i \in T$, there is $j, k \in T$ such that

$$j + i = 0 \tag{18}$$

and

$$k + j = 0 \tag{19}$$

by associativity:

$$k + (j + i) = (k + j) + i \tag{20}$$

$$k + 0 = 0 + i \tag{21}$$

$$k = i. \tag{22}$$

Thus, by (19) and (22), j is a right inverse of i. Hence, by Proposition 9, $TS(L)$ has complete past.
Proof in the converse direction runs similarly, *mutatis mutandis* (hint: use first Proposition 9 in place of Corollary 3 and, second, Corollary 3 in place of Proposition 9).
Q.E.D.

As a consequence of Proposition 8 and Proposition 10, any reversible time system is logically reversible and it has complete past – a property which goes under the name of *complete logical reversibility*.

Definition 10: (Completely logically reversible dynamical system)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.
DS_L *is completely logically reversible* iff for any $t \in T$, g^t is bijective.

In time systems, complete logical reversibility turns out to be equivalent to reversibility, as shown below.

Proposition 11: Let $TS(L) = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.
$TS(L)$ is reversible iff $TS(L)$ is completely logically reversible.

Proof: Suppose $TS(L)$ is reversible; by Proposition 8, $TS(L)$ is logically reversible and, by Proposition 10, $TS(L)$ has complete past. Hence, by Definition 10, $TS(L)$ is logically reversible.
Conversely, suppose $TS(L)$ is completely logically reversible; then, by Definition 10, $TS(L)$ has complete past. Hence, by Proposition 10, $TS(L)$ is reversible. *Q.E.D.*

2.3. *Time Invertibility*

The strongest form of reversibility a dynamical system may display is that of time invertibility, defined below.

Definition 10: (Time invertible dynamical system)
Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.
DS_L *is time invertible* iff L is a group.

In the general case, it is possible to show that any time invertible dynamical system is completely logically reversible, with complete past, strictly reversible, and reversible, but all four converse implications are false. By contrast, in the special case of time systems, time invertibility turns out to be equivalent to reversibility (see Proposition 12 below), and thus to strict reversibility (by Corollary 3), complete past (by Proposition 10), and complete logical reversibility (by Proposition 11).

Proposition 12: Let $TS(L) = (I, (\iota^t)_{t \in T})$ be the time system of a monoid $L = (T, +)$ with identity 0.
$TS(L)$ is reversible iff $TS(L)$ is time invertible.

Proof: suppose $TS(L)$ is reversible; then, by Corollary 3, Proposition 10 and Proposition 9, for any $i \in T$ there is $j, k \in T$ such that
$j + i = 0$ (23)
$i + k = 0$ (24)
and, by associativity

$$j = j + 0 = j + (i + k) = (j + i) + k = 0 + k = k \tag{25}$$

Thus, L is a group and, by Definition 11, $TS(L)$ is time invertible.
Conversely, suppose $TS(L)$ is time invertible; then, by Definition 11, for any $t \in T$ and any $i \in I = T$ there is $-t \in T$ such that

$$\iota^{-t}(\iota^t(i)) = (-t + t) + i = 0 + i = i \tag{26}$$

hence, by Definition 6, $TS(L)$ is reversible. *Q.E.D.*

The main implication of Proposition 12 is that we are left with just two distinct concepts of reversible time – namely, reversibility in the proper sense and logical reversibility, where the first one entails the latter. In the proper sense, reversibility of time is tantamount to time being algebraically represented by a group. In the merely logical sense, by contrast, time is reversible just in that, for any instant i, if an instant j that precedes i of a duration t exists, then j is unique, so that j could in principle be retrieved from knowledge of i alone.

Topological analyses of time almost invariably come to the conclusion that time is a differentiable manifold diffeomorphic to the set R of real numbers, which has the rich algebraic structure of a field, and thus of an additive group. In the light of Proposition 12, it is then no surprise that those proposing such analyses typically deny time to possess any intrinsic direction, earning its apparent directionality from that of the physical processes it hosts.

3 Conclusion

In the first section we have shown that, in virtue of its possessing algebraic properties, time is endowed with intrinsic dynamics. A further analysis has revealed that some of the dynamical properties a time system may display – such as reversibility, strict reversibility, complete past, and complete logical reversibility – not only essentially depend on its algebraic structure, but are in fact equivalent to the requirement that such a structure be a group. Since properties such as the directionality of a dynamical system depend by definition on the algebraic properties of its time model, it follows that they must also depend on the dynamical properties of the corresponding time system.

In particular we saw that, in the case of time systems, reversibility is a very strong property implying, among the others, time invertibility. As a consequence, if the time system of a monoid L is reversible – or, equivalently, strictly reversible, with complete past, or completely logically reversible – then all dynamical systems on L must be time invertible, and hence reversible, strictly reversible, with complete past and completely logically reversible. The converse, however, is obviously not true: reversible dynamical systems may well possess irreversible time systems, dynamical systems with complete past may possess time systems with incomplete past, and so on. We may thus conclude that the internal dynamics of time is generally predominant over the dynamics of the processes that take place in it.

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